

The homomorphisms between scalar generalized Verma modules associated to maximal parabolic subalgebras

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Abstract

Let \mathfrak{g} be a finite-dimensional simple Lie algebra over \mathbb{C} . We classify the homomorphisms between \mathfrak{g} -modules induced from one-dimensional modules of maximal parabolic subalgebras.¹

§ 0. Introduction

In this article, we consider the existence problem of homomorphisms between generalized Verma modules, which are induced from one dimensional representations (such generalized Verma modules are called scalar, cf. [Boe 1985]). Our main result is the classification of the homomorphisms between scalar generalized Verma modules with respect to the maximal parabolic subalgebras.

A sufficient condition for the existence of the homomorphisms between Verma modules is given by [Verma 1968]. Bernstein, I. M. Gelfand, and S. I. Gelfand proved the condition of Verma is also a necessary condition. ([Bernstein-Gelfand-Gelfand 1975])

Later, Lepowsky studied the problem for the generalized Verma modules. In particular, Lepowsky ([Lepowsky 1975a]) solved the existence problem of nontrivial homomorphisms between scalar generalized Verma modules associated to the parabolic subalgebras which are the complexifications of the minimal parabolic subalgebras of real rank one simple Lie algebras (so-called the real rank one case).

Lepowsky also obtained a sufficient condition for the existence of the homomorphisms between scalar generalized Verma modules associated to the complexification of the minimal parabolic subalgebras of (not necessarily rank one) real semisimple Lie algebras. His condition is quite similar to that of Verma and he conjectured it is also a sufficient condition in the setting of complexified minimal parabolic algebras ([Lepowsky 1975b]).

Boe ([Boe 1985]) solved the existence problem in the case of parabolic subalgebras whose nilradical is commutative (so-called the Hermitian symmetric case).

The existence problem for maximal parabolic algebras is, in principle, reduced to the Kazhdan-Lusztig algorithm. Casian and Collingwood ([Casian-Collingwood 1987]) proposed a direct

¹Keywords: generalized Verma module, differential invariant
 AMS Mathematical Subject Classification: 22E47, 17B10

method of computing the Kazhdan-Lusztig data involving the generalized Verma modules. Applying the Kazhdan-Lusztig algorithm works very well in some cases. The structures of the (not necessarily scalar) generalized Verma modules in the real rank one case and in the Hermitian symmetric case are studied precisely ([Boe-Collingwood 1985], [Boe-Enright-Shelton 1988], [Collingwood-Irving-Shelton 1988]). Boe and Collingwood ([Boe-Collingwood 1990]) studied the case that all the (not necessarily scalar) generalized Verma modules are multiplicity free. The cases treated by Boe and Collingwood is more general than the real rank one case and the Hermitian symmetric case (so-called the multiplicity free case). In that case, they studied the structures of the (not necessarily scalar) generalized Verma modules precisely. In particular, they solved the existence problem of the non-trivial homomorphisms for regular integral infinitesimal characters in the multiplicity free case.

However, I surmise it is not easy to give the explicit answer to the existence problem by the Kazhdan-Lusztig algorithm in the general setting. Our central dogma is “*Consider the most singular parameter, then everything turns to be easy.*” Our approach to the problem consists of the following three main ingredients.

(1) The translation principle

The translation principle has a long history. In [Vogan 1988], Vogan proposed an idea on translation principle in order to establish the irreducibility of a discrete series representation of a semisimple symmetric space in some case. His idea is extremely useful for the study of the existence problem. Depending Vogan’s idea, we formulated a version of translation principle in [Matumoto 1993] Proposition 2.2.3. In some cases, this enable us to reduce the existence of a non-trivial homomorphism to the most singular case in which the problem is often trivial.

(2) Jantzen’s irreducibility criterion

Applying a version of the translation principle, we can often reduce the nonexistence of a non-trivial homomorphism to the irreducibility of a particular generalized Verma module. In [Jantzen 1977], Jantzen gave a sufficient and necessary condition for the irreducibility of a generated Verma module. His result is extremely useful for our purpose and we can establish the nonexistence of nontrivial homomorphisms in many cases.

(3) The Kazhdan-Lusztig theory ([Kazhdan-Lusztig 1979], [Brylinski-Kashiwara 1981], [Beilinson-Bernstein 1993])

Although we do not compute Kazhdan-Lusztig polynomials, the existence of the Kazhdan-Lusztig algorithm plays an important role in our approach. The point is that the definition of the Kazhdan-Lusztig polynomials only depends on Coxeter systems. In some cases, this enable us to reduce the problem to that of a different maximal parabolic subalgebra of a different simple Lie algebra, which is easier than the original problem.

In the most of the cases we can solve the existence problem by the above three ideas. However, in some cases we need extra arguments.

This article consists of five sections.

We fix notations and introduce some fundamental material in §1.

In §2, we introduce sufficient conditions for the existence problem.

In §3, we treat the case of the classical algebras. The type A case is in the Hermitian symmetric case. So, we only consider the case of the type B,C, and D. The main theorems are Theorem 3.2.1, 3.2.2, and 3.2.3.

In §4, we treated the case of the exceptional algebras.

In §5, using a comparison result, we explain how to construct a homomorphism between scalar generalized Verma modules associated to a general parabolic subalgebra from a homomorphism

between scalar generalized Verma modules associated to a maximal parabolic subalgebra. We call such a homomorphism an elementary homomorphism.

I would like to propose:

Working Hypothesis *An arbitrary nontrivial homomorphism between scalar generalized Verma modules is a composition of elementary homomorphisms.*

The working hypothesis in the case of the Verma modules is nothing but the result of Bernstein-Gelfand-Gelfand. The first statement of the Lepowsky conjecture ([Lepowsky 1975b] Conjecture 6.13) means that, in the case of complexified minimal parabolic subalgebras of real semisimple Lie algebras, the above working hypothesis is affirmative.

The result of this article solves the existence of edge-of-wedge type embeddings in the case of the maximal parabolic subgroups of complex reductive groups ([Matumoto 2003]).

Acknowledgements

During the proceeding of this study, I participated in the “Representation Theory of Lie Groups” at the Institute for Mathematical Sciences (IMS) the National University of Singapore. It is my pleasure to thank their warm hospitality.

§ 1. Notations and Preliminaries

1.1 General notations

In this article, we use the following notations and conventions.

As usual we denote the complex number field, the real number field, the ring of (rational) integers, and the set of non-negative integers by \mathbb{C} , \mathbb{R} , \mathbb{Z} , and \mathbb{N} respectively. $\frac{1}{2}\mathbb{N}$ means the set $\{\frac{n}{2} \mid n \in \mathbb{N}\}$, and $\frac{1}{2} + \mathbb{N}$ means the set $\{\frac{1}{2} + n \mid n \in \mathbb{N}\}$. We denote by \emptyset the empty set. For any (non-commutative) \mathbb{C} -algebra R , “ideal” means “2-sided ideal”, “ R -module” means “left R -module”, and sometimes we denote by 0 (resp. 1) the trivial R -module $\{0\}$ (resp. \mathbb{C}). Often, we identify a (small) category and the set of its objects. Hereafter “dim” means the dimension as a complex vector space, and “ \otimes ” (resp. Hom) means the tensor product over \mathbb{C} (resp. the space of \mathbb{C} -linear mappings), unless we specify. For a complex vector space V , we denote by V^* the dual vector space. For $a, b \in \mathbb{C}$, “ $a \leq b$ ” means that $a, b \in \mathbb{R}$ and $a \leq b$. We denote by $A - B$ the set theoretical difference. $\text{card}A$ means the cardinality of a set A .

1.2 Notations for reductive Lie algebras

Let \mathfrak{g} be a complex reductive Lie algebra, $U(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} , and \mathfrak{h} a Cartan subalgebra of \mathfrak{g} . We denote by Δ the root system with respect to $(\mathfrak{g}, \mathfrak{h})$. We fix some positive root system Δ^+ and let Π be the set of simple roots. Let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ and let $\langle \cdot, \cdot \rangle$ be a non-degenerate invariant bilinear form on \mathfrak{g} . For $w \in W$, we denote by $\ell(w)$ the length of w as usual. We also denote the inner product on \mathfrak{h}^* which is induced from the above form by the same symbols $\langle \cdot, \cdot \rangle$. For $\alpha \in \Delta$, we denote by s_α the reflection in W with respect to α . We denote by w_0 the longest element of W . For $\alpha \in \Delta$, we define the coroot $\check{\alpha}$ by $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$, as usual. We call $\lambda \in \mathfrak{h}^*$ is dominant (resp. anti-dominant), if $\langle \lambda, \check{\alpha} \rangle$ is not a negative (resp. positive) integer, for each $\alpha \in \Delta^+$. We call $\lambda \in \mathfrak{h}^*$ regular, if $\langle \lambda, \alpha \rangle \neq 0$, for each $\alpha \in \Delta$. We denote by P the integral weight lattice, namely $P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$.

If $\lambda \in \mathfrak{h}^*$ is contained in P , we call λ an integral weight. We define $\rho \in P$ by $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Put $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h} \ [H, X] = \alpha(H)X\}$, $\mathfrak{u} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{u}$. Then \mathfrak{b} is a Borel subalgebra of \mathfrak{g} . We denote by Q the root lattice, namely \mathbb{Z} -linear span of Δ . We also denote by Q^+ the linear combination of Π with non-negative integral coefficients. For $\lambda \in \mathfrak{h}^*$, we denote by W_λ the integral Weyl group. Namely,

$$W_\lambda = \{w \in W \mid w\lambda - \lambda \in Q\}.$$

We denote by Δ_λ the set of integral roots.

$$\Delta_\lambda = \{\alpha \in \Delta \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}\}.$$

It is well-known that W_λ is the Weyl group for Δ_λ . We put $\Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda$. This is a positive system of Δ_λ . We denote by Π_λ the set of simple roots for Δ_λ^+ and denote by Φ_λ the set of reflection corresponding to the elements in Π_λ . So, $(W_\lambda, \Phi_\lambda)$ is a Coxeter system. We denote by Q_λ the integral root lattice, namely $Q_\lambda = \mathbb{Z}\Delta_\lambda^+$ and put $Q_\lambda^+ = \mathbb{N}\Pi_\lambda$.

Next, we fix notations for a parabolic subalgebra (which contains \mathfrak{b}). Hereafter, through this article we fix an arbitrary subset Θ of Π . Let $\bar{\Theta}$ be the set of the elements of Δ which are written by linear combinations of elements of Θ over \mathbb{Z} . Put $\mathfrak{a}_\Theta = \{H \in \mathfrak{h} \mid \forall \alpha \in \Theta \ \alpha(H) = 0\}$, $\mathfrak{l}_\Theta = \mathfrak{h} + \sum_{\alpha \in \bar{\Theta}} \mathfrak{g}_\alpha$, $\mathfrak{n}_\Theta = \sum_{\alpha \in \Delta^+ \setminus \bar{\Theta}} \mathfrak{g}_\alpha$, $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta + \mathfrak{n}_\Theta$. Then \mathfrak{p}_Θ is a parabolic subalgebra of \mathfrak{g} which contains \mathfrak{b} . Conversely, for an arbitrary parabolic subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$, there exists some $\Theta \subseteq \Pi$ such that $\mathfrak{p} = \mathfrak{p}_\Theta$. We denote by W_Θ the Weyl group for $(\mathfrak{l}_\Theta, \mathfrak{h})$. W_Θ is identified with a subgroup of W generated by $\{s_\alpha \mid \alpha \in \Theta\}$. We denote by w_Θ the longest element of W_Θ . Using the invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$, we regard \mathfrak{a}_Θ^* as a subspace of \mathfrak{h}^* . It is known that there is a unique nilpotent (adjoint) orbit (say $\mathcal{O}_{\mathfrak{p}_\Theta}$) whose intersection with \mathfrak{n}_Θ is Zarisky dense in \mathfrak{n}_Θ . $\mathcal{O}_{\mathfrak{p}_\Theta}$ is called the Richardson orbit with respect to \mathfrak{p}_Θ . We denote by $\bar{\mathcal{O}}_{\mathfrak{p}_\Theta}$ the closure of $\mathcal{O}_{\mathfrak{p}_\Theta}$ in \mathfrak{g} . Put $\rho_\Theta = \frac{1}{2}(\rho - w_\Theta \rho)$ and $\rho^\Theta = \frac{1}{2}(\rho + w_\Theta \rho)$. Then, $\rho^\Theta \in \mathfrak{a}_\Theta^*$.

1.3 Generalized Verma modules

Define

$$\begin{aligned} P_\Theta^{++} &= \{\lambda \in \mathfrak{h}^* \mid \forall \alpha \in \Theta \ \langle \lambda, \check{\alpha} \rangle \in \{1, 2, \dots\}\} \\ {}^\circ P_\Theta^{++} &= \{\lambda \in \mathfrak{h}^* \mid \forall \alpha \in \Theta \ \langle \lambda, \check{\alpha} \rangle = 1\} \end{aligned}$$

We easily have

$${}^\circ P_\Theta^{++} = \{\rho_\Theta + \mu \mid \mu \in \mathfrak{a}_\Theta^*\}.$$

For $\mu \in \mathfrak{h}^*$ such that $\mu + \rho \in P_\Theta^{++}$, we denote by $\sigma_\Theta(\mu)$ the irreducible finite-dimensional \mathfrak{l}_Θ -representation whose highest weight is μ . Let $E_\Theta(\mu)$ be the representation space of $\sigma_\Theta(\mu)$. We define a left action of \mathfrak{n}_Θ on $E_\Theta(\mu)$ by $X \cdot v = 0$ for all $X \in \mathfrak{n}_\Theta$ and $v \in E_\Theta(\mu)$. So, we regard $E_\Theta(\mu)$ as a $U(\mathfrak{p}_\Theta)$ -module.

For $\mu \in P_\Theta^{++}$, we define a generalized Verma module ([Lepowsky 1977]) as follows.

$$M_\Theta(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} E_\Theta(\mu - \rho).$$

For all $\lambda \in \mathfrak{h}^*$, we write $M(\lambda) = M_\emptyset(\lambda)$. $M(\lambda)$ is called a Verma module. For $\mu \in P_\Theta^{++}$, $M_\Theta(\mu)$ is a quotient module of $M(\mu)$. Let $L(\mu)$ be the unique highest weight $U(\mathfrak{g})$ -module with the

highest weight $\mu - \rho$. Namely, $L(\mu)$ is a unique irreducible quotient of $M(\mu)$. For $\mu \in \mathbf{P}_\Theta^{++}$, the canonical projection of $M(\mu)$ to $L(\mu)$ is factored by $M_\Theta(\mu)$.

$\dim E_\Theta(\mu - \rho) = 1$ if and only if $\mu \in {}^\circ\mathbf{P}_\Theta^{++}$. If $\mu \in {}^\circ\mathbf{P}_\Theta^{++}$, we call $M_\Theta(\mu)$ a scalar generalized Verma module.

For a finitely generated $U(\mathfrak{g})$ -module V , we denote by $\text{Dim}(V)$ (resp. $c(V)$) the Gelfand-Krillov dimension (resp. the multiplicity) of V . (See [Vogan 1978]). We easily see $\text{Dim}(M_\Theta(\mu)) = \dim \mathfrak{n}_\Theta$ and $c(M_\Theta(\mu)) = \dim E_\Theta(\mu - \rho)$.

The following result is one of the fundamental results on the existence problem of homomorphisms between scalar generalized Verma modules.

Theorem 1.3.1. ([Lepowsky 1976])

Let $\mu, \nu \in {}^\circ\mathbf{P}_\Theta^{++}$.

- (1) $\dim \text{Hom}_{U(\mathfrak{g})}(M_\Theta(\mu), M_\Theta(\nu)) \leq 1$.
- (2) Any non-zero homomorphism of $M_\Theta(\mu)$ to $M_\Theta(\nu)$ is injective.

Hence, the existence problem of homomorphisms between scalar generalized Verma modules is reduced to the following problem.

Problem Let $\mu, \nu \in {}^\circ\mathbf{P}_\Theta^{++}$. When is $M_\Theta(\mu) \subseteq M_\Theta(\nu)$?

1.4 Homomorphisms associated with Duflo involutions

Herafter we assume $\Theta \subseteq \Pi$, $\mu \in \mathbf{P}_\Theta^{++}$ and μ is dominant and regular. Then, we easily have $w_\Theta w_0 \mu \in \mathbf{P}_\Theta^{++}$ and $M_\Theta(w_\Theta w_0 \mu)$ is irreducible. Here, we consider the following problem

Problem 1.4.1. When is $M_\Theta(w_\Theta w_0 \mu) \hookrightarrow M_\Theta(\mu)$?

Concerning to Problem 1.4.1, a necessary and sufficient condition is known.

Theorem 1.4.2. ([Matumoto 1993]) Let $\Theta \subseteq \Pi$ and $\mu \in \mathbf{P}_\Theta^{++}$. If μ is dominant and regular, then the following two conditions (1) and (2) are equivalent.

- (1) $M_\Theta(w_\Theta w_0 \mu) \hookrightarrow M_\Theta(\mu)$.
- (2) $w_\Theta w_0$ is a Duflo involution for the Coxeter system (W_μ, Φ_μ) .

In particular, the answer of Problem 2.1.1 only depends on the Coxeter system (W_μ, Φ_μ) . (In fact, this fact is a conclusion of the Kazhdan-Lusztig conjecture.) We can find a complex reductive Lie algebra whose Weyl group (with a set of the simple reflections) are isomorphic to (W_μ, Φ_μ) . So, we can deduce Problem 1.4.1 to the following special case.

Problem 1.4.3. When is $M_\Theta(w_\Theta w_0 \rho) \hookrightarrow M_\Theta(\rho)$?

We also remark that the following easy fact.

Lemma 1.4.4. Let $\Theta \subseteq \Pi$ and $\lambda \in {}^\circ\mathbf{P}_\Theta^{++}$ such that $w_\Theta w_0 = w_0 w_\Theta$ and $w_\Theta w_0 \in W_\lambda$. We denote by w'_0 be the longest element of W_λ with respect to Π_λ . Then, $w_0 = w'_0$ and $w_\Theta w'_0 = w'_0 w_\Theta$.

1.5 Translation principle and its application

We denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. It is well-known that $Z(\mathfrak{g})$ acts on $M(\lambda)$ by the Harish-Chandra homomorphism $\chi_\lambda : Z(\mathfrak{g}) \rightarrow \mathbb{C}$ for all λ . $\chi_\lambda = \chi_\mu$ if and only if there exists some $w \in W$ such that $\lambda = w\mu$. We denote by \mathbf{Z}_λ the kernel of χ_λ in $Z(\mathfrak{g})$. Let M be a $U(\mathfrak{g})$ -module and $\lambda \in \mathfrak{h}^*$. We say that M has an infinitesimal character λ if and only if $Z(\mathfrak{g})$ acts on M by χ_λ . We say that M has a generalized infinitesimal character λ if and only if for any $v \in M$ there is some positive integer n such that $\mathbf{Z}_\lambda^n v = 0$. We say M is locally $Z(\mathfrak{g})$ -finite, if and only if for any $v \in M$ we have $\dim Z(\mathfrak{g})v < \infty$. We denote by \mathcal{M}_{Zf} (cf. [Bernstein-Gelfand 1980]) the category of $Z(\mathfrak{g})$ -finite $U(\mathfrak{g})$ -modules. We also denote by $\mathcal{M}[\lambda]$ the category of $U(\mathfrak{g})$ -modules with generalized infinitesimal character λ . Then, from the Chinese remainder theorem, we have a direct sum of abelian categories $\mathcal{M}_{Zf} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathcal{M}[\lambda]$. We denote by P_λ the projection functor from \mathcal{M}_{Zf} to $\mathcal{M}[\lambda]$. For $\mu \in \mathbf{P}$, we denote by V_μ the irreducible finite-dimensional $U(\mathfrak{g})$ -module with an extreme weight μ . Let $\mu, \lambda \in \mathfrak{h}^*$ satisfy $\mu - \lambda \in \mathbf{P}$. Let M be an object of $\mathcal{M}[\lambda]$. Then, from a result of Kostant we have that $M \otimes V_{\mu-\lambda}$ is an object of \mathcal{M}_{Zf} . So, we can define translation functor T_λ^μ from $\mathcal{M}[\lambda]$ to $\mathcal{M}[\mu]$ as follows.

$$T_\lambda^\mu(M) = P_\mu(M \otimes V_{\mu-\lambda}).$$

The translation functors are exact.

We put

$$W(\Theta) = \{w \in W \mid w\Theta = \Theta\}.$$

Then, $W(\Theta)$ is a subgroup of W . Moreover, $w\rho_\Theta = \rho_\Theta$ and $w_\Theta w = ww_\Theta$ hold for all $w \in W(\Theta)$ and $W(\Theta)$ preserves \mathfrak{a}_Θ^* . In particular, $W(\Theta) \subseteq W_{\rho_\Theta}$.

We say that $\lambda \in \mathfrak{a}_\Theta^*$ is strongly Θ -antidominant if and only if $\langle \lambda, \alpha \rangle \leq 0$ for all $\alpha \in \mathbf{Q}^+ \cap \mathbf{Q}_{\rho_S + \lambda}$. (Cf. [Matumoto 1993])

Next, we consider the images of generalized Verma modules under certain translation functors.

Lemma 1.5.1. (Cf. [Matumoto 1993] Lemma 1.2.3) Assume that $\mu, \lambda \in \mathfrak{a}_\Theta^*$ are strongly Θ -antidominant and that $\lambda - \mu$ is dominant and integral. Let $w \in W(\Theta)$. Then, we have

$$T_{-\rho_\Theta + \mu}^{-\rho_\Theta + \lambda}(M_\Theta(\rho_\Theta + w\mu)) = M_\Theta(\rho_\Theta + w\lambda).$$

§ 2. Sufficient conditions

For almost all $(\mathfrak{g}, \mathfrak{h}, \Delta^+, \Theta)$, the necessary and sufficient condition given in Theorem 1.4.2 is hard to check. So, we consider sufficient conditions, which we can check easily.

2.1 A sufficient condition

We fix the notations for characters. (Cf. [Dixmier 1977] 7.5.1, [Knapp 2002] V. 6) Let $\mathbb{C}^{\mathfrak{h}^*}$ be the \mathbb{C} -vector space of all functions from \mathfrak{h}^* to \mathbb{C} . For $f \in \mathbb{C}^{\mathfrak{h}^*}$, we define $\text{supp}(f) = \{\lambda \in \mathfrak{h}^* \mid f(\lambda) \neq 0\}$. For $\lambda \in \mathfrak{h}^*$, we define e^λ the member of $\mathbb{C}^{\mathfrak{h}^*}$, that is 1 at λ and 0 elsewhere. Let $\mathbb{C} < \mathfrak{h}^* >$ be the set of all $f \in \mathbb{C}^{\mathfrak{h}^*}$ such that $\text{supp}(f)$ is contained in the union of a finite number of sets $\nu_i - Q^+$ with each ν_i in \mathfrak{h}^* . We introduce the structure of \mathbb{C} -algebra on $\mathbb{C}^{\mathfrak{h}^*}$ as in [Knapp 2002] (5.65).

Let V be a $U(\mathfrak{g})$ -module. For $\lambda \in \mathfrak{h}^*$, we define the weight space with respect to λ as follows.

$$V_\lambda = \{v \in V \mid \forall H \in \mathfrak{h} \ Hv = \lambda(H)v\}$$

We say that V has a character if V is the direct sum of its weight spaces under \mathfrak{h} and if $\dim V_\lambda \leq \infty$ for all $\lambda \in \mathfrak{h}^*$. In this case, the character is

$$[V] = \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda) e^\lambda.$$

For example, for $\lambda \in \mathcal{P}_\Theta^{++}$, the following formula is well-known.

$$[M_\Theta(\lambda)] = D^{-1} \sum_{w \in W_\Theta} (-1)^{\ell(w)} e^{w\lambda}.$$

Here, we denote by D the Weyl denominator, namely $D = e^\rho \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})$. In particular, we have $[M(\lambda)] = D^{-1} e^\lambda$.

We put $\mathbf{B}_{st} = \{[M(w\rho)] \mid w \in W\}$ and denote by \mathcal{C} the subspace of $\mathbb{C}^{\mathfrak{h}^*}$ spanned by \mathbf{B}_{st} . We also put $\mathbf{B}_{irr} = \{[L(w\rho)] \mid w \in W\}$. Then, \mathbf{B}_{st} and \mathbf{B}_{irr} are bases of \mathcal{C} . We identify the group algebra $\mathbb{C}[W]$ with \mathcal{C} via the correspondence $W \ni w \rightsquigarrow [M(w\rho)] \in \mathbf{B}_{st}$. We can introduce a W -module structure on \mathcal{C} identifying \mathcal{C} and the right regular representation on $\mathbb{C}[W]$. We consider the equivalence relations $\overset{L}{\sim}$ and $\overset{R}{\sim}$ on W defined in [Kazhdan-Lusztig 1979]. The representation theoretic meanings of these equivalence relations are as follows.

Theorem 2.1.1. ([Joseph 1977], [Vogan 1980]) *Let $x, y \in W$. Then we have*

- (1) $x \overset{L}{\sim} y$ if and only if $\text{Ann}_{U(\mathfrak{g})}(L(xw_0\rho)) = \text{Ann}_{U(\mathfrak{g})}(L(yw_0\rho))$.
- (2) $x \overset{R}{\sim} y$ if and only if there exist some finite-dimensional $U(\mathfrak{g})$ -modules E_1 and E_2 such that $L(xw_0\rho)$ and $L(yw_0\rho)$ are irreducible constituents of $E_1 \otimes L(yw_0\rho)$ and $E_2 \otimes L(xw_0\rho)$, respectively.

For $x \in W$, we denote by V_x^R the \mathbb{C} -vector space with a basis $B_x = \{[L(yw_0\rho)] \mid y \overset{R}{\sim} x\}$. If we identify V_x^R with a subquotient of \mathcal{C} appropriately, we may regard V_x^R a W -module. (For example, see [Barbasch-Vogan 1983]). V_x^R is called a right cell representation.

We denote by \mathcal{H} the space of W -harmonic polynomials on \mathfrak{h}^* , which can be regarded as a W -module in a usual manner (cf. [Vogan 1978]).

We quote:

Theorem 2.1.2. ([Vogan 1978], [Joseph 1980a 1980b])

For $x \in W$, there is a W -homomorphism ϕ_x of V_x^R to \mathcal{H} satisfying the following conditions.

- (1) *The image $\phi_x(V_x^R) \subseteq \mathcal{H}$ is an irreducible representation.*
- (2) *The image $\phi_x(V_x^R)$ is the special representation corresponding to the unique open dense nilpotent orbit in the associated variety of $\text{Ann}_{U(\mathfrak{g})}(L(xw_0\rho))$ via the Springer correspondence. In particular, $\phi_{w_\Theta}(V_{w_\Theta}^R)$ is the special representation corresponding to the Richardson orbit $\mathcal{O}_{\mathfrak{p}_\Theta}$.*
- (3) *For $y, z \in W$ such that $y \overset{R}{\sim} x \overset{R}{\sim} z$, $y \overset{L}{\sim} z$ if and only if $\phi_x([L(yw_0\rho)])$ and $\phi_x([L(zw_0\rho)])$ are proportional to each other.*

Now, we state a sufficient condition.

Proposition 2.1.3. *Assume that $\Theta \subseteq \Pi$ satisfy $w_\Theta w_0 = w_0 w_\Theta$. Moreover, we assume that $V_{w_\Theta}^R$ is irreducible as a W -module. Then, we have $M_\Theta(w_\Theta w_0 \rho) \subseteq M_\Theta(\rho)$.*

Proof. We put $I = \text{Ann}_{U(\mathfrak{g})}(M_\Theta(w_\Theta w_0 \rho))$. I is primitive, since $M_\Theta(w_\Theta w_0 \rho)$ is irreducible. Since $w_\Theta w_0 = w_0 w_\Theta$, we have $w_0 w_\Theta \in W(S)$. So, from [Borho-Jantzen 1977] 4.10 Corollary, we have $I = \text{Ann}_{U(\mathfrak{g})}(M_\Theta(\rho))$. Since $c(M_\Theta(\rho)) = 1$, $M_\Theta(\rho)$ has a unique irreducible constituent of maximal Gelfand-Killirov dimension (say $L(\sigma\rho)$). Here, σ is an element of W . (In fact we we easy to prove $L(\sigma\rho)$ is the unique irreducible submodule of $M_\Theta(\rho)$.) Assuming that $\sigma \neq w_\Theta w_0$, we shall deduce a contradiction.

When we regard a $U(\mathfrak{g})$ -module E as a $U(\mathfrak{p}_\Theta)$ -module, we write it by $E|_{\mathfrak{p}_\Theta}$. Since $M_\Theta(\mu) \otimes E \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} (E_\Theta(\mu - \rho) \otimes E|_{\mathfrak{p}_\Theta})$ holds, we easily see that $\sigma w_0 \stackrel{R}{\sim} w_\Theta$.

From [Borho-Kraft 1976] 3.6, we see $I = \text{Ann}_{U(\mathfrak{g})}(L(\sigma\rho))$. From Theorem 1.6.1, we have $w_\Theta \stackrel{L}{\sim} \sigma w_0$.

Hence, from Theorem 1.6.2, ϕ_{w_Θ} is not injective. This contradicts our assumption that $V_{w_\Theta}^R$ is irreducible. Q.E.D.

The multiplicity of a special representation $\phi_x(V_x^R)$ in a right cell V_x^R is one. Moreover, any irreducible constituents in the right cell V_x^R belongs to the same family (see [Lusztig 1984] p78) as $\phi_x(V_x^R)$. So, we have:

Corollary 2.1.4. *Assume that $\Theta \subseteq \Pi$ satisfy $w_\Theta w_0 = w_0 w_\Theta$ and that the family of the special representation corresponding to the Richardson orbit $\mathcal{O}_{\mathfrak{p}_\Theta}$ does not contain any other element. Then, we have $M_\Theta(w_\Theta w_0 \rho) \subseteq M_\Theta(\rho)$.*

We denote by $\stackrel{LR}{\sim}$ the equivalence relation on W generated by $\stackrel{L}{\sim}$ and $\stackrel{R}{\sim}$. The following result is well-known and follows from Theorem 2.1.1.

Corollary 2.1.5. *Let $x, y \in W$ be such that $x \stackrel{LR}{\sim} y$. Then $\text{Dim}(L(xw_0\rho)) = \text{Dim}(L(yw_0\rho))$.*

From [Barbasch-Vogan 1983] Corollary 2.24 implies that $x \stackrel{LR}{\sim} y$ if and only if $xw_0 \stackrel{LR}{\sim} yw_0$. Hence, we have:

Lemma 2.1.6. *Let $x, y \in W$ be such that $x \stackrel{LR}{\sim} y$. Then $\text{Dim}(L(x\rho)) = \text{Dim}(L(y\rho))$.*

2.2 Maximal parabolic subalgebras

Hereafter we fix $\alpha \in \Pi$. Put $\Theta^\alpha = \Pi - \{\alpha\}$. $\mathfrak{a}_{\Theta^\alpha}$ is one-dimensional and spanned by ρ^{Θ^α} . Moreover, we have

$${}^\circ P_{\Theta^\alpha}^{++} = \{\rho_{\Theta^\alpha} + t\rho^{\Theta^\alpha} \mid t \in \mathbb{C}\}.$$

We denote by ω_α the fundamental weight corresponding to α . For any $\beta \in \Theta^\alpha = \Pi - \{\alpha\}$, we have $\langle \beta, \rho^{\Theta^\alpha} \rangle = 0$. Hence there exists some $d_\alpha \in \mathbb{R}$ such that $d_\alpha \omega_\alpha = \rho^{\Theta^\alpha}$. Since $2\rho^{\Theta^\alpha}$ is integral, we have $d_\alpha \in \frac{1}{2}\mathbb{N}$.

For simplicity, for $t \in \mathbb{C}$, we write $M_{\Theta^\alpha}[t]$ for $M_{\Theta^\alpha}(\rho_{\Theta^\alpha} + t\omega_\alpha)$. We have:

Lemma 2.2.1. *Let s and t be distinct complex numbers such that $M_{\Theta^\alpha}[s] \subseteq M_{\Theta^\alpha}[t]$. Then, $s = -t$.*

Proof. Since $M_{\Theta^\alpha}[s]$ and $M_{\Theta^\alpha}[t]$ have the same infinitesimal character, there exists some $w \in W$ such that $\rho_{\Theta^\alpha} + t\omega_\alpha = w(\rho_{\Theta^\alpha} + s\omega_\alpha)$. Hence, $\langle \rho_{\Theta^\alpha} + t\omega_\alpha, \rho_{\Theta^\alpha} + t\omega_\alpha \rangle = \langle \rho_{\Theta^\alpha} + s\omega_\alpha, \rho_{\Theta^\alpha} + s\omega_\alpha \rangle$. From $\langle \omega_\alpha, \rho_{\Theta^\alpha} \rangle = 0$, we have $t^2 = s^2$. So, $s = -t$. \square

We easily have:

Lemma 2.2.2. (1) If $w_{\Theta^\alpha}w_0 = w_0w_{\Theta^\alpha}$, then $w_{\Theta^\alpha}w_0(\rho_{\Theta^\alpha} + t\omega_\alpha) = \rho_{\Theta^\alpha} - t\omega_\alpha$ for all $t \in \mathbb{C}$.
(2) If $w_{\Theta^\alpha}w_0 \neq w_0w_{\Theta^\alpha}$, then $w_{\Theta^\alpha}w_0(\rho_{\Theta^\alpha} + t\omega_\alpha) = \rho_{\Theta^\alpha} - t\omega_\alpha$ if and only if $t = d_\alpha$.

We put

$$c_\alpha = \min\{c \in \mathbb{R} \mid 2c\omega_\alpha \in \mathbb{Q}^+\}.$$

Clearly $2c_\alpha$ is a positive integer.

Lemma 2.2.3. If $w_{\Theta^\alpha}w_0 = w_0w_{\Theta^\alpha}$, then either $c_\alpha = 1$ or $c_\alpha = \frac{1}{2}$.

Proof. We have only to show $c_\alpha \leq 1$. We may assume \mathfrak{g} is simple. If \mathfrak{g} is a simple Lie algebra of the type other than A_n , D_{2n+1} , and E_6 , then the exponent of \mathbb{Q}/\mathbb{P} is 1 or 2. So, in this case $2\omega_\alpha \in \mathbb{Q}^+$ for any $\alpha \in \Pi$. For the case of the type A_n , D_{2n+1} , or E_6 , we can check $2\omega_\alpha \in \mathbb{Q}^+$ under the assumption $w_{\Theta^\alpha}w_0 = w_0w_{\Theta^\alpha}$ by the case-by-case analysis. \square

If $M_{\Theta^\alpha}[-t] \subseteq M_{\Theta^\alpha}[t]$ holds, then $2t\omega_\alpha = (\rho_{\Theta^\alpha} + t\omega_\alpha) - (\rho_{\Theta^\alpha} - t\omega_\alpha)$ is in \mathbb{Q}^+ . So, we have:

Corollary 2.2.4. If $w_{\Theta^\alpha}w_0 = w_0w_{\Theta^\alpha}$ and if $M_{\Theta^\alpha}[-t] \subseteq M_{\Theta^\alpha}[t]$ holds, then $t \in \frac{1}{2}\mathbb{N}$. Moreover, if $c_\alpha = 1$, then $M_{\Theta^\alpha}[-t] \hookrightarrow M_{\Theta^\alpha}[t]$ implies $t \in \mathbb{N}$.

Definition 2.2.5. If $t \in \frac{1}{2}\mathbb{N}$ and $\rho_{\Theta^\alpha} + t\omega_\alpha$ is not integral, we say $\rho_{\Theta^\alpha} + t\omega_\alpha$ is half-integral.

We examine behavior of the translation functors in the setting of this subsection.

First, Lemma 1.5.1 and the exactness of the translation functor imply:

Lemma 2.2.6. Assume $w_{\Theta^\alpha}w_0 = w_0w_{\Theta^\alpha}$. Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$ be such that $t - n \geq 0$. Then, $M_{\Theta^\alpha}[-t] \subseteq M_{\Theta^\alpha}[t]$ implies $M_{\Theta^\alpha}[-t + n] \subseteq M_{\Theta^\alpha}[t - n]$.

From the translation principle, we also have:

Lemma 2.2.7. Assume $w_{\Theta^\alpha}w_0 = w_0w_{\Theta^\alpha}$. Let $t \in \mathbb{R}$ be such that $\rho_{\Theta^\alpha} + t\omega_\alpha$ is dominant regular. Let $n \in \mathbb{N}$. Then, $M_{\Theta^\alpha}[-t] \subseteq M_{\Theta^\alpha}[t]$ implies $M_{\Theta^\alpha}[-t - n] \subseteq M_{\Theta^\alpha}[t + n]$.

In case $\rho_{\Theta^\alpha} + t\omega_\alpha$ is not dominant regular, the corresponding statement to Lemma 1.7.6 is not necessarily correct. In fact, we need an extra assumption.

Let G be a complex connected reductive Lie group, whose Lie algebra is \mathfrak{g} . Let P_Θ be the parabolic subgroup of G corresponding to \mathfrak{p}_Θ . We consider the generalized flag variety $X_\Theta = G/P_\Theta$. Since the holomorphic cotangent bundle T^*X_Θ has a natural symplectic structure, we can construct the moment map $m_\Theta : T^*X_\Theta \rightarrow \mathfrak{g}^*$. Using $\langle \cdot, \cdot \rangle$, we identify \mathfrak{g} and \mathfrak{g}^* . Then, we regard the moment map as a surjective map of T^*X_Θ to the closure of the Richardson orbit $\mathcal{O}_{\mathfrak{p}_\Theta}$.

We easily see the number $N_1(P_\Theta)$ defined in [Hesselink 1978] 1.4 Step3 is the degree of the moment map $m_\Theta : T^*X_\Theta \rightarrow \overline{\mathcal{O}_{\mathfrak{p}_\Theta}}$. So, $N_1(P_\Theta) = 1$ if and only if m_Θ is birational. For classical Lie algebras, $N_1(P_\Theta)$ is obtained in [Hesselink 1978] 7.1 Theorem.

Lemma 2.2.8. ([Matumoto 1993] Proposition 2.2.3) Assume $w_{\Theta^\alpha}w_0 = w_0w_{\Theta^\alpha}$ and the moment map $m_\Theta : T^*X_\Theta \rightarrow \overline{\mathcal{O}_{\mathfrak{p}_{\Theta^\alpha}}}$ is birational. Let $n \in \mathbb{N}$. Then, $M_{\Theta^\alpha}[-t] \subseteq M_{\Theta^\alpha}[t]$ implies $M_{\Theta^\alpha}[-t - n] \subseteq M_{\Theta^\alpha}[t + n]$.

Since $M_{\Theta^\alpha}[0] \subseteq M_{\Theta^\alpha}[0]$, we have

Corollary 2.2.9. ([Matumoto 1993] Corollary 2.2.4)

Assume $w_{\Theta^\alpha}w_0 = w_0w_{\Theta^\alpha}$ and the moment map $m_\Theta : T^*X_{\Theta^\alpha} \rightarrow \overline{\mathcal{O}_{\mathfrak{p}_{\Theta^\alpha}}}$ is birational. Then, we have $M_{\Theta^\alpha}[-n] \subseteq M_{\Theta^\alpha}[n]$ for all $n \in \mathbb{N}$.

Next, we introduce Jantzen's criterion for the irreducibility of a generalized Verma module. For any $\lambda \in \mathfrak{h}^*$, we define an element of $\mathbb{C} < \mathfrak{h}^* >$ as follows.

$$\Upsilon_{\Theta^\alpha}(\lambda) = D^{-1} \sum_{w \in W_{\Theta^\alpha}} (-1)^{\ell(w)} e^{w\lambda}.$$

Of course, for $\lambda \in \mathfrak{P}_{\Theta^\alpha}^{++}$, we have $\Upsilon_{\Theta^\alpha}(\lambda) = [M_{\Theta^\alpha}(\lambda)]$.

Put $\Delta_{\Theta^\alpha} = \mathbb{Z}\Theta^\alpha \cap \Delta$ and $(\Delta^{\Theta^\alpha})^+ = \Delta^+ - \Delta_{\Theta^\alpha}$. We immediately have:

Corollary 2.2.10.

- (1) If $\lambda \in \mathfrak{h}^*$ satisfies $\langle \beta, \lambda \rangle = 0$ for some $\beta \in \Delta_{\Theta^\alpha}$, then $\Upsilon_{\Theta^\alpha}(\lambda) = 0$.
- (2) For $\lambda \in \mathfrak{h}^*$ and $w \in W_{\Theta^\alpha}$, we have $\Upsilon_{\Theta^\alpha}(\lambda) - (-1)^{\ell(w)} \Upsilon_{\Theta^\alpha}(w\lambda) = 0$.

The following result is a special case of [Jantzen 1977] Satz 3.

Theorem 2.2.11. (Jantzen)

For $\lambda \in \mathfrak{P}_{\Theta^\alpha}^{++}$, the following (1) and (2) are equivalent.

- (1) $M_{\Theta^\alpha}(\lambda)$ is irreducible.
- (2) We have

$$\sum_{\substack{\beta \in (\Delta^{\Theta^\alpha})^+ \\ \langle \lambda, \beta^\vee \rangle \in \mathbb{N} - \{0\}}} \Upsilon_{\Theta^\alpha}(s_\beta \lambda) = 0.$$

Remark The above statement is slightly different from that in Jantzen's paper. However, we consider maximal parabolic subalgebras. In this case, we can easily see the above condition (2) is equivalent to the condition of Jantzen.

§ 3. Classical Lie algebras

Throughout this section, n means a positive integer such that $n \geq 2$ (resp. $n \geq 4$) whenever we consider the simple Lie algebra of the type B_n or C_n (resp. D_n).

3.1 The root systems

We retain the notations in §1 and §2.

(B_n type) We consider the root system Δ for $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$. Then we can choose an orthonormal basis e_1, \dots, e_n of \mathfrak{h}^* such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}.$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{e_i \mid 1 \leq i \leq n\}.$$

If we put $\alpha_i = e_i - e_{i+1}$ ($1 \leq i < n$) and $\alpha_n = e_n$, then $\Pi = \{\alpha_1, \dots, \alpha_n\}$.

For simplicity, we write Θ^k for Θ^{α_k} for $1 \leq k \leq n$. Then \mathfrak{l}_{Θ^k} is isomorphic to $\mathfrak{gl}(k, \mathbb{C}) \times \mathfrak{so}(2(n-k)+1, \mathbb{C})$. Since w_0 is contained in the center of W , we have $w_0 w_{\Theta^k} = w_{\Theta^k} w_0$ for any $1 \leq k \leq n$.

We write ω_k , c_k , and d_k for ω_{α_k} , c_{α_k} , and d_{α_k} , respectively. Then, for $1 \leq k < n$, we have $\omega_k = e_1 + \dots + e_k$, $c_k = \frac{1}{2}$, and $d_k = n - \frac{k}{2}$. If $k = n$, then $\omega_n = \frac{1}{2}(e_1 + \dots + e_n)$, $c_n = 1$, and $d_n = n$.

Assume k is even or $k = n$. Then, $\rho_{\Theta^k} + t\omega_k$ is integral if and only if $t \in \mathbb{Z}$.

Assume k is odd and $1 \leq k < n$. Then, $\rho_{\Theta^k} + t\omega_k$ is integral if and only if $t - \frac{1}{2} \in \mathbb{Z}$.

(C_n type)

We consider the root system Δ for $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. Then we can choose an orthonormal basis e_1, \dots, e_n of \mathfrak{h}^* such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2e_i \mid 1 \leq i \leq n\}.$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{2e_i \mid 1 \leq i \leq n\}.$$

If we put $\alpha_i = e_i - e_{i+1}$ ($1 \leq i < n$) and $\alpha_n = 2e_n$, then $\Pi = \{\alpha_1, \dots, \alpha_n\}$.

For simplicity, we write Θ^k for Θ^{α_k} for $1 \leq k \leq n$. Then \mathfrak{l}_{Θ^k} is isomorphic to $\mathfrak{gl}(k, \mathbb{C}) \times \mathfrak{sp}(n-k, \mathbb{C})$. Since w_0 is contained in the center of W , we have $w_0 w_{\Theta^k} = w_{\Theta^k} w_0$ for any $1 \leq k \leq n$.

We write ω_k , c_k , and d_k for ω_{α_k} , c_{α_k} , and d_{α_k} , respectively. Then, for $1 \leq k \leq n$ we have $\omega_k = e_1 + \dots + e_k$ and $d_k = n - \frac{k-1}{2}$. If k is even (resp. odd), then $c_k = \frac{1}{2}$ (resp. $c_k = 1$).

Assume k is odd. Then, $\rho_{\Theta^k} + t\omega_k$ is integral if and only if $t \in \mathbb{Z}$.

Assume k is even. Then, $\rho_{\Theta^k} + t\omega_k$ is integral if and only if $t - \frac{1}{2} \in \mathbb{Z}$.

(D_n type)

We consider the root system Δ for $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. Then we can choose an orthonormal basis e_1, \dots, e_n of \mathfrak{h}^* such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}.$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq n\}.$$

If we put $\alpha_i = e_i - e_{i+1}$ ($1 \leq i < n$) and $\alpha_n = e_{n-1} + e_n$, then $\Pi = \{\alpha_1, \dots, \alpha_n\}$.

For simplicity, we write Θ^k for Θ^{α_k} for $1 \leq k \leq n$. Since the case of $k = n-1$ is essentially same as the case of $k = n$, since $\mathfrak{p}_{\Theta^{n-1}}$ and \mathfrak{p}_{Θ^n} are conjugate under an automorphism of \mathfrak{g} . So, when we consider the type D case, we omit the case of $k = n-1$.

Then, \mathfrak{l}_{Θ^k} is isomorphic to $\mathfrak{gl}(k, \mathbb{C}) \times \mathfrak{so}(2(n-k), \mathbb{C})$. We have $w_0 w_{\Theta^k} = w_{\Theta^k} w_0$ for any $1 \leq k < n$. However, $w_0 w_{\Theta^n} = w_{\Theta^n} w_0$ if and only if n is even.

We write ω_k , c_k , and d_k for ω_{α_k} , c_{α_k} , and d_{α_k} , respectively. Then, for $1 \leq k < n-1$, we have $\omega_k = e_1 + \dots + e_k$ and $d_k = n - \frac{k+1}{2}$. If k is even (resp. odd) and $k \neq n$, then $c_k = \frac{1}{2}$ (resp. $c_k = 1$).

For $k = n$, we have $\omega_n = \frac{1}{2}(e_1 + \dots + e_n)$, $c_n = 1$, and $d_n = n$. If n is even (resp. odd), then $c_n = 1$ (resp. $c_n = 2$).

Assume k is odd or $k = n$. Then, $\rho_{\Theta^k} + t\omega_k$ is integral if and only if $t \in \mathbb{Z}$.

Assume k is even and $1 \leq k < n$. Then, $\rho_{\Theta^k} + t\omega_k$ is integral if and only if $t - \frac{1}{2} \in \mathbb{Z}$.

3.2 Statements of the main results

Here, we describe the existence of homomorphisms between scalar generalized Verma modules with respect to the maximal parabolic subalgebras of classical Lie algebras. For simple Lie algebras of type A, the answer is given in [Boe 1985]. So, we treat the case of the types of B, C, and D.

Theorem 3.2.1. (B_n -type)

Consider the case of $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$. Let k be a positive integer such that $k \leq n$.

- (1) We consider the case that $3k < 2n+1$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \mathbb{N}$.
- (2) We consider the case that $3k \geq 2n+1$.
 - (2a) Assume k is odd and $k \neq n$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.
 - (2b) Assume n is odd. Then, $M_{\Theta^n}[-t] \subseteq M_{\Theta^n}[t]$ if and only if $t \in \mathbb{N}$.
 - (2c) Assume k is even. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t = 0$.

Remark The case of $k = 1$ is due to [Lepowsky 1975a]. The cases of $k = 1, 2, 3, n$ are in the multiplicity free case in [Boe-Collingwood 1990].

Theorem 3.2.2. (C_n -type)

Consider the case of $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. Let k be a positive integer such that $k \leq n$.

- (1) We consider the case that $3k \leq 2n$.
 - (1a) Assume k is odd. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t = 0$.
 - (1b) Assume k is even. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.
- (2) We consider the case that $3k > 2n$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \mathbb{N}$.

Remark The case of $k = 2$ is due to [Lepowsky 1975a]. The case of $k = n$ is due to [Boe 1985]. The cases of $k = 1, 2, 3, n$ are in the multiplicity free case in [Boe-Collingwood 1990].

Theorem 3.2.3. (D_n -type)

Consider the case of $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. Let k be a positive integer such that $k \leq n-2$ or $k = n$.

- (1) We consider the case that $3k < 2n$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \mathbb{N}$.
- (2) We consider the case that $3k \geq 2n$.
 - (2a) Assume k is odd. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t = 0$.
 - (2b) Assume k is even and $k \neq n$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.
 - (2c) Assume n is even. Then, $M_{\Theta^n}[-t] \subseteq M_{\Theta^n}[t]$ if and only if $t \in \mathbb{N}$.

Remark The case of $k = 1$ is due to [Lepowsky 1975a]. The case of $k = n$ is due to [Boe 1985]. The cases of $k = 1, 2, n$ are in the multiplicity free case in [Boe-Collingwood 1990].

We give proofs of the above theorems in the subsequent sections.

3.3 Richardson orbits

We consider a partition $\pi = (p_1, \dots, p_k)$ of a positive integer m such that $0 < p_1 \leq p_2 \leq \dots \leq p_k$ and $p_1 + p_2 + \dots + p_k = m$. We put $\pi[i] = \text{card}\{j \mid p_j = i\}$ for any positive integer i . Let $\{h_1, \dots, h_r\} = \{i \in \mathbb{N} \mid i > 0, \pi[i] > 0\}$. Then, we write $h_1^{\pi[h_1]} \dots h_r^{\pi[h_r]}$ for π . For example, $5 \cdot 3^2 \cdot 1^5$ means $(1, 1, 1, 1, 1, 3, 3, 5)$.

Type B_n The nilpotent orbits in $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ are parametrized by partitions π of $2n+1$ such that, for any even number $2i$, $\pi[2i]$ is even. (For example see [Carter 1985] p394) From [Collingwood-McGovern 1993] 7 and [Hesselink 1978] 7.1, we have:

Lemma 3.3.1. *Let $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$.*

- (1) *If $3k < 2n+1$, then the Richardson orbit $\mathcal{O}_{\mathfrak{p}_{\Theta^k}}$ corresponds to the partition $3^k \cdot 1^{2n+1-3k}$. In this case, the moment map m_{Θ^k} is birational.*
- (2) *If $3k \geq 2n+1$ and k is odd, then the Richardson orbit $\mathcal{O}_{\mathfrak{p}_{\Theta^k}}$ corresponds to the partition $3^{2n+1-2k} \cdot 2^{3k-2n-1}$. In this case, the moment map m_{Θ^k} is birational.*
- (3) *If $3k \geq 2n+1$ and k is even, then the Richardson orbit $\mathcal{O}_{\mathfrak{p}_{\Theta^k}}$ corresponds to the partition $3^{2n+1-2k} \cdot 2^{3k-2n-2} \cdot 1^2$.*

From Corollary 2.2.9, and Lemma 3.3.1, we have:

Lemma 3.3.2. *Let $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$. Assume that $3k \leq 2n+1$ or k is odd. Then, we have $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ for all $t \in \mathbb{N}$.*

Type \mathbf{C}_n The nilpotent orbits in $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ are parametrized by partitions π of $2n$ such that, for any odd number $2i+1$, $\pi[2i+1]$ is even. (For example see [Carter 1985] p394) From [Collingwood-McGovern 1993] 7 and [Carter 1985] Chapter 13, we have:

Lemma 3.3.3. *Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$.*

- (1) *If $3k \leq 2n$ and k is even, then the Richardson orbit $\mathcal{O}_{\mathfrak{p}_{\Theta^k}}$ corresponds to the partition $3^k \cdot 1^{2n-3k}$. In this case the moment map m_{Θ^k} is birational.*
- (2) *If $3k \leq 2n$ and k is odd, then the Richardson orbit $\mathcal{O}_{\mathfrak{p}_{\Theta^k}}$ corresponds to the partition $3^{k-1} \cdot 2^2 \cdot 1^{2n-3k-1}$.*
- (3) *If $3k > 2n$, then the Richardson orbit $\mathcal{O}_{\mathfrak{p}_{\Theta^k}}$ corresponds to the partition $3^{2n-2k} \cdot 2^{3k-2n}$. In this case, the moment map m_{Θ^k} is birational.*

From Corollary 2.2.9, and Lemma 3.3.3, we have:

Lemma 3.3.4. *Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. Assume that $3k > 2n$ or k is even. Then, we have $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ for all $t \in \mathbb{N}$.*

Type \mathbf{D}_n We can associate a nilpotent orbit in $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ with a partition $\pi = (p_1, \dots, p_k)$ of $2n$ such that, for any even number $2i$, $\pi[2i]$ is even. If there is at least one odd number in p_1, \dots, p_k , there is a unique nilpotent orbit associated with (p_1, \dots, p_k) . The exceptional orbits are so-called very even nilpotent orbits. If p_1, \dots, p_k are all even, there are two nilpotent orbits associated with the partition (p_1, \dots, p_k) . (For example see [Carter 1985] p395) From [Collingwood-McGovern 1993] 7 and [Hesselink 1978] 7.1, we have:

Lemma 3.3.5. *Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and let $1 \leq k \leq n-2$ or $k = n$.*

- (1) *If $3k \leq 2n$, then the Richardson orbit $\mathcal{O}_{\mathfrak{p}_{\Theta^k}}$ corresponds to the partition $3^k \cdot 1^{2n-3k}$. In this case, the moment map m_{Θ^k} is birational.*
- (2) *If $3k > 2n$ and k is odd, then the Richardson orbit $\mathcal{O}_{\mathfrak{p}_{\Theta^k}}$ corresponds to the partition $3^{2n-2k} \cdot 2^{3k-2n-1} \cdot 1^2$.*
- (3) *If $3k > 2n$ and k is even, then the Richardson orbit $\mathcal{O}_{\mathfrak{p}_{\Theta^k}}$ corresponds to the partition $3^{2n-2k} \cdot 2^{3k-2n}$. In this case, the moment map m_{Θ^k} is birational.*

From Corollary 2.2.9, and Lemma 3.3.5, we have:

Lemma 3.3.6. *Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. Assume that $3k \leq 2n$ or k is even. Then, we have $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ for all $t \in \mathbb{N}$.*

3.4 Existence results via comparison

In this subsection, we prove the following result.

Lemma 3.4.1.

- (1) Assume that $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ and that k is an even positive integer such that $3k \leq 2n$. Then, we have $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ for all $t \in \frac{1}{2} + \mathbb{N}$.
- (2) Assume that $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ and that k is an odd positive integer such that $3k \geq 2n+1$ and $k \neq n$. Then, we have $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ for all $t \in \frac{1}{2} + \mathbb{N}$.

Proof We prove (1). We can prove (2) in a similar way.

Let k be an even positive integer such that $3k \leq 2n$. At first, we consider the case that $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$. In this case $\rho^{\Theta^k} = d_k \omega_k = (n - \frac{k}{2})\omega_k$ from 3.1. Therefore, $\rho = \rho_{\Theta^k} + (n - \frac{k}{2})\omega_k$ and $w_{\Theta^k} w_0 \rho = \rho_{\Theta^k} - (n - \frac{k}{2})\omega_k$. Since $n - \frac{k}{2} \in \mathbb{N}$, Lemma 3.3.2 implies $M_{\Theta^k}(w_{\Theta^k} w_0 \rho) \subseteq M_{\Theta^k}(\rho)$. From Theorem 1.4.2, $w_{\Theta^k} w_0$ is a Duflo involution in the Weyl group for $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$. However, the Weyl group of the B_n -type and that of the C_n -type are isomorphic to each other as a Coxeter system. Since the notion of the Duflo involutions only depends on the structure of the Coxeter system, we see that as an element of the Weyl group for $\mathfrak{sp}(n, \mathbb{C})$, $w_{\Theta^k} w_0$ is a Duflo involution. From Theorem 1.4.2 implies $M_{\Theta^k}(w_{\Theta^k} w_0 \rho) \subseteq M_{\Theta^k}(\rho)$ holds for the case that $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. Namely, $M_{\Theta^k}[-d_k] \subseteq M_{\Theta^k}[d_k]$. In this case, $d_k = n - \frac{n-1}{2} \in \frac{1}{2} + \mathbb{N}$. So, from Lemma 2.2.6 and Lemma 2.2.7, we have $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ for all $t \in \frac{1}{2} + \mathbb{N}$. Q.E.D.

3.5 Irreducibility of a right cell

In this subsection, we treat the remaining case that nontrivial homomorphisms exist. Namely, we show:

Lemma 3.5.1. Assume that $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and that k is an even positive integer such that $3k \geq 2n$, $k \leq n$, and $k \neq n-1$. Then, we have $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ for all $t \in \frac{1}{2} + \mathbb{N}$.

Proof. If $k = n$, then \mathfrak{n}_{Θ^k} is abelian. This case is treated in [Boe 1985]. So, we assume that k is an even positive integer such that $3k > 2n$, $k < n-1$. Put $s = \frac{k}{2}$. As in [Carter 1985] p376, we can associate an irreducible representation of the Weyl group of the type D_n with a so-called symbol. A symbol is a pair of strictly increasing sequence of non-negative integers of the same length. We identify two symbols;

$$\begin{pmatrix} \lambda_1, \dots, \lambda_k \\ \mu_1, \dots, \mu_k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mu_1, \dots, \mu_k \\ \lambda_1, \dots, \lambda_k \end{pmatrix}.$$

As in [Carter 1985] 13.3, we can associate the partition $3^{2n-2k} \cdot 2^{3k-2n}$ to a pair of partitions as follows.

$$\begin{pmatrix} 1, 2, 3, \dots, 3s-n, 3s-n+2, 3s-n+3, 3s-n+4, \dots, s+2 \\ 1, 2, 3, \dots, 3s-n, 3s-n+1, 3s-n+2, 3s-n+3, \dots, s+1 \end{pmatrix}$$

This is the symbol associated with the special representation (say π_k) corresponding to the Richardson orbit $\mathcal{O}_{\mathfrak{p}_{\Theta^k}}$ via the Springer correspondence. (Lusztig, Shoji)

As in [Carter 1985] 13.2, two irreducible representation with symbols:

$$\begin{pmatrix} \lambda_1, \dots, \lambda_k \\ \mu_1, \dots, \mu_k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda'_1, \dots, \lambda'_k \\ \mu'_1, \dots, \mu'_k \end{pmatrix}$$

are contained in the same family if and only if $\{\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_k\} = \{\lambda'_1, \dots, \lambda'_k, \mu'_1, \dots, \mu'_k\}$. So, we easily see the family of π_k consists of only one element. From Corollary 2.1.4, we have $M_{\Theta^k}(w_{\Theta^k} w_0 \rho) \subseteq M_{\Theta^k}(\rho)$. In this case, $d_k = n - \frac{k+1}{2} \in \frac{1}{2} + \mathbb{N}$. So, from Lemma 2.2.6 and Lemma 2.2.7, we have $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ for all $t \in \frac{1}{2} + \mathbb{N}$. Q.E.D.

Remark We can prove Lemma 3.4.1 in a similar way to Lemma 3.4.2.

3.6 Irreducibility of some generalized Verma modules

We have:

Lemma 3.6.1.

- (1) Assume that $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ and that k is an even positive integer such that $k < n$. Then, $M_{\Theta^k}(\frac{1}{2})$ is irreducible.
- (2) Assume that $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ and that n is even. Then, $M_{\Theta^n}(1)$ is irreducible.
- (3) Assume that $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and that k is an even positive integer such that $3k < 2n$. Then, $M_{\Theta^k}(\frac{1}{2})$ is irreducible.
- (4) Assume that $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ and that k is an odd positive integer such that $3k < 2n+1$. Then, $M_{\Theta^k}(\frac{1}{2})$ is irreducible.
- (5) Assume that $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ and that k is an even positive integer such that $3k > 2n$. Then, $M_{\Theta^k}(\frac{1}{2})$ is irreducible.
- (6) Assume that $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. Then, $M_{\Theta^1}(1)$ is irreducible.

Proof. For $2 \leq r \leq k$, we put $r^* = k + 2 - r$.

First, we prove (1). We put $s = \frac{k}{2}$. Then, we see

$$\rho_{\Theta^k} + \frac{1}{2}\omega_k = \sum_{i=1}^{2s} (s-i+1)e_i + \sum_{j=2s+1}^n (n-j+\frac{1}{2})e_j.$$

We easily see

$$\begin{aligned} & \left\{ \beta \in (\Delta^{\Theta^k})^+ \mid \langle \rho_{\Theta^k} + \frac{1}{2}\omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\} \right\} \\ &= \{e_i \mid 1 \leq i \leq s\} \cup \{e_i + e_j \mid 1 \leq i < j \leq 2s, 2s+2 > i+j\}. \end{aligned}$$

For $2 \leq i \leq s$, $\langle e_i - e_{2s-i+2}, s_{e_i}(\rho_{\Theta^k} + \frac{1}{2}\omega_k) \rangle = 0$. Hence, Corollary 2.2.10 implies $\Upsilon_{\Theta^k}(s_{e_i}(\rho_{\Theta^k} + \frac{1}{2}\omega_k)) = 0$.

Next, we assume that $1 \leq i < j \leq 2s$ and that $2s+2 > i+j$. Since $i < j$, $2s+2 > i+j$ implies $i \leq s$. If $2 \leq i$, then we see $j \neq i^*$ and $\langle e_j - e_{i^*}, s_{e_i+e_j}(\rho_{\Theta^k} + \frac{1}{2}\omega_k) \rangle = 0$. So, Corollary 2.2.10 implies $\Upsilon_{\Theta^k}(s_{e_i+e_j}(\rho_{\Theta^k} + \frac{1}{2}\omega_k)) = 0$. If $i = 1$ and $j \neq s+1$, then $j \neq j^* \neq 1$ and $\langle e_1 - e_{j^*}, s_{e_i+e_j}(\rho_{\Theta^k} + \frac{1}{2}\omega_k) \rangle = 0$. So, Corollary 2.2.10 implies $\Upsilon_{\Theta^k}(s_{e_i+e_j}(\rho_{\Theta^k} + \frac{1}{2}\omega_k)) = 0$.

Hence, we have

$$\sum_{\substack{\beta \in (\Delta^{\Theta^k})^+ \\ \langle \rho_{\Theta^k} + \frac{1}{2}\omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\}}} \Upsilon_{\Theta^k}(s_\beta(\rho_{\Theta^k} + \frac{1}{2}\omega_k)) = \Upsilon_{\Theta^k}(s_{e_1}(\rho_{\Theta^k} + \frac{1}{2}\omega_k)) + \Upsilon_{\Theta^k}(s_{e_1+e_{s+1}}(\rho_{\Theta^k} + \frac{1}{2}\omega_k)).$$

Since $\langle e_{s+1}, \rho_{\Theta^k} + \frac{1}{2}\omega_k \rangle = 0$, we have $s_{e_1}(\rho_{\Theta^k} + \frac{1}{2}\omega_k) = s_{e_1-e_{s+1}}s_{e_1+e_{s+1}}(\rho_{\Theta^k} + \frac{1}{2}\omega_k)$. Hence from Corollary 2.2.10, we have

$$\sum_{\substack{\beta \in (\Delta^{\Theta^k})^+ \\ \langle \rho_{\Theta^k} + \frac{1}{2}\omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\}}} \Upsilon_{\Theta^k}(s_\beta(\rho_{\Theta^k} + \frac{1}{2}\omega_k)) = 0.$$

From Jantzen's criterion (Theorem 2.2.11), $M_{\Theta^k}[\frac{1}{2}]$ is irreducible. So, we get (1).

(2) is proved in the same way as (1).

Next, we prove (3).

We put $s = \frac{k}{2}$. Then, we see

$$\rho_{\Theta^k} + \frac{1}{2}\omega_k = \sum_{i=1}^{2s} (s-i+1)e_i + \sum_{j=2s+1}^n (n-j)e_j.$$

We easily see

$$\begin{aligned} & \left\{ \beta \in (\Delta^{\Theta^k})^+ \mid \langle \rho_{\Theta^k} + \frac{1}{2}\omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\} \right\} \\ &= \{e_i + e_j \mid 1 \leq i < j \leq 2s, 2s+2 > i+j\} \\ & \quad \cup \{e_i \pm e_j \mid 1 \leq i \leq 2s < j \leq n, (s-i+1) \pm (n-j) > 0\}. \end{aligned}$$

Since $3k < 2n$, we have $2s < n-s$. Let $\beta \in (\Delta^{\Theta^k})^+$ be such that $\langle \rho_{\Theta^k} + \frac{1}{2}\omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\}$. Moreover, we assume β is neither $e_1 + e_{s+1}$ nor $e_1 + e_{n-s}$. Then, we can easily see there exists some $\gamma \in \Delta_{\Theta^k}$ such that $\langle s_\beta(\rho_{\Theta^k} + \frac{1}{2}\omega_k), \gamma^\vee \rangle = 0$. From Corollary 2.2.10, we have

$$\sum_{\substack{\beta \in (\Delta^{\Theta^k})^+ \\ \langle \rho_{\Theta^k} + \frac{1}{2}\omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\}}} \Upsilon_{\Theta^k}(s_\beta(\rho_{\Theta^k} + \frac{1}{2}\omega_k)) = \Upsilon_{\Theta^k}(s_{e_1+e_{s+1}}(\rho_{\Theta^k} + \frac{1}{2}\omega_k)) + \Upsilon_{\Theta^k}(s_{e_1+e_{n-s}}(\rho_{\Theta^k} + \frac{1}{2}\omega_k)).$$

However, $s_{e_1+e_{s+1}}(\rho_{\Theta^k} + \frac{1}{2}\omega_k) = s_{e_1-e_{s+1}}s_{e_{n-s}-e_n}s_{e_{n-s}+e_n}s_{e_1+e_{n-s}}(\rho_{\Theta^k} + \frac{1}{2}\omega_k)$ holds. So, Corollary 2.2.10 implies

$$\sum_{\substack{\beta \in (\Delta^{\Theta^k})^+ \\ \langle \rho_{\Theta^k} + \frac{1}{2}\omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\}}} \Upsilon_{\Theta^k}(s_\beta(\rho_{\Theta^k} + \frac{1}{2}\omega_k)) = 0.$$

From Jantzen's criterion (Theorem 2.2.11), $M_{\Theta^k}[\frac{1}{2}]$ is irreducible. So, we get (3).

Next, we prove (4). We put $s = \frac{k-1}{2}$. Then, we see

$$\rho_{\Theta^k} + \frac{1}{2}\omega_k = \sum_{i=1}^{2s+1} \left(s-i+\frac{3}{2}\right)e_i + \sum_{j=2s+2}^n \left(n-j+\frac{1}{2}\right)e_j.$$

We easily see

$$\begin{aligned} & \left\{ \beta \in \left(\Delta^{\Theta^k} \right)^+ \mid \left\langle \rho_{\Theta^k} + \frac{1}{2} \omega_k, \beta^\vee \right\rangle \in \mathbb{N} - \{0\} \right\} \\ &= \{ e_i + e_j \mid 1 \leq i < j \leq 2s+1, 2s+3 > i+j \} \\ & \cup \left\{ e_i \pm e_j \mid 1 \leq i \leq 2s < j \leq n, \left(s-i + \frac{3}{2} \right) \pm \left(n-j + \frac{1}{2} \right) > 0 \right\}. \end{aligned}$$

Since $3k < 2n+1$, we have $s + \frac{1}{2} \leq n - 2s - \frac{3}{2}$. Let $\beta \in \left(\Delta^{\Theta^k} \right)^+$ be such that $\langle \rho_{\Theta^k} + \frac{1}{2} \omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\}$. Moreover, we assume β is neither e_1 nor $e_1 + e_{n-s}$. Then, we can easily see there exists some $\gamma \in \Delta_{\Theta^k}$ such that $\langle s_\beta(\rho_{\Theta^k} + \frac{1}{2} \omega_k), \gamma^\vee \rangle = 0$. From Corollary 2.2.10, we have

$$\sum_{\substack{\beta \in \left(\Delta^{\Theta^k} \right)^+ \\ \langle \rho_{\Theta^k} + \frac{1}{2} \omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\}}} \Upsilon_{\Theta^k}(s_\beta(\rho_{\Theta^k} + \frac{1}{2} \omega_k)) = \Upsilon_{\Theta^k}(s_{e_1}(\rho_{\Theta^k} + \frac{1}{2} \omega_k)) + \Upsilon_{\Theta^k}(s_{e_1+e_{n-s}}(\rho_{\Theta^k} + \frac{1}{2} \omega_k)).$$

However, $s_{e_1}(\rho_{\Theta^k} + \frac{1}{2} \omega_k) = s_{e_{n-s}} s_{e_1+e_{n-s}}(\rho_{\Theta^k} + \frac{1}{2} \omega_k)$ holds. So, Corollary 2.2.10 implies

$$\sum_{\substack{\beta \in \left(\Delta^{\Theta^k} \right)^+ \\ \langle \rho_{\Theta^k} + \frac{1}{2} \omega_k, \beta^\vee \rangle \in \mathbb{N} - \{0\}}} \Upsilon_{\Theta^k}(s_\beta(\rho_{\Theta^k} + \frac{1}{2} \omega_k)) = 0.$$

From Jantzen's criterion (Theorem 2.2.11), $M_{\Theta^k}[\frac{1}{2}]$ is irreducible. So, we get (4).

(5) and (6) is due to [Gyoja 1994] p394. They are proved similarly. Q.E.D.

Lemma 3.6.1 and Lemma 2.2.6 imply:

Lemma 3.6.2. (1) Assume that $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ and that k is an even positive integer such that $k < n$. Then, $M_{\Theta^k}[-t - \frac{1}{2}] \not\subseteq M_{\Theta^k}[t + \frac{1}{2}]$ for all $t \in \mathbb{N}$.

(2) Assume that $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ and that n is even. Then, $M_{\Theta^k}[-t] \not\subseteq M_{\Theta^k}[t]$ for all $t \in \mathbb{N}$.

(3) Assume that $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and that k is an even positive integer such that $3k < 2n$. Then, $M_{\Theta^k}[-t - \frac{1}{2}] \not\subseteq M_{\Theta^k}[t + \frac{1}{2}]$ for all $t \in \mathbb{N}$.

(4) Assume that $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ and that k is an odd positive integer such that $3k < 2n+1$. Then, $M_{\Theta^k}[-t - \frac{1}{2}] \not\subseteq M_{\Theta^k}[t + \frac{1}{2}]$ for all $t \in \mathbb{N}$.

(5) Assume that $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ and that k is an even positive integer such that $3k > 2n$. Then, $M_{\Theta^k}[-t - \frac{1}{2}] \not\subseteq M_{\Theta^k}[t + \frac{1}{2}]$ for all $t \in \mathbb{N}$.

(6) Assume that $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. Then, $M_{\Theta^1}[-t] \not\subseteq M_{\Theta^1}[t]$ for all $t \in \mathbb{N}$.

3.7 Nonexistence results for the remaining cases

First, we assume that $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$, n is odd, and $k = n$. This case is treated in [Boe 1985]. In fact, his result contains:

Lemma 3.7.1. (Boe) Assume $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and n is odd. Then, $M_{\Theta^n}[-t] \subseteq M_{\Theta^n}[t]$ if and only if $t = 0$.

Hereafter we do not consider the above case. Therefore, from 2.1, we have $w_{\Theta^k} w_0 = w_0 w_{\Theta^k}$. The results in 3.3-3.6 and Corollary 2.2.4 imply that Theorem 3.2.1-3.2.3 is reduced to the following lemma.

Lemma 3.7.2.

- (1) Assume that $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ and that k is an even positive integer such that $3k > 2n+1$ and $k < n$. Then, $M_{\Theta^k}[-t-1] \not\subseteq M_{\Theta^k}[t+1]$ for all $t \in \mathbb{N}$.
- (2) Assume that $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ and that k is an odd positive integer such that $3k \leq 2n$ and $1 < k$. Then, $M_{\Theta^k}[-t-1] \not\subseteq M_{\Theta^k}[t+1]$ for all $t \in \mathbb{N}$.
- (3) Assume that $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and that k is an odd positive integer such that $3k \geq 2n$. Then, $M_{\Theta^k}[-t-1] \not\subseteq M_{\Theta^k}[t+1]$ for all $t \in \mathbb{N}$.

Proof. From Lemma 2.2.6, we have only to show that $M_{\Theta^k}[-1] \not\subseteq M_{\Theta^k}[1]$.

We put $\Omega = W \cdot (\rho_{\Theta^k} + \omega_k) \cap \mathcal{P}_{\Theta^k}^{++}$. Then, in the settings of (1), (2), and (3) above, Ω consists of four elements (say $\lambda_1, \dots, \lambda_4$). We can write $\lambda_1 = \rho_{\Theta^k} + \omega_k$ and $\lambda_4 = \rho_{\Theta^k} - \omega_k$. The remaining two elements are as follows.

If $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ and if $k = 2s$ is an even positive integer such that $3k > 2n+1$ and $k < n$, then we may write:

$$\begin{aligned} \lambda_2 &= \left(s + \frac{1}{2}\right) e_1 + \sum_{i=2}^{2s} \left(s + \frac{1}{2} - i\right) e_i + \sum_{j=2s+1}^n \left(n + \frac{1}{2} - j\right) e_j, \\ \lambda_3 &= \sum_{i=1}^{2s-1} \left(s + \frac{1}{2} - i\right) e_i + \left(-\frac{1}{2} - s\right) e_{2s} + \sum_{j=2s+1}^n \left(n + \frac{1}{2} - j\right) e_j. \end{aligned}$$

If $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ and if $k = 2s+1$ is an odd positive integer such that $3k \leq 2n$ and $1 < k$, then we may write:

$$\begin{aligned} \lambda_2 &= (s+1) e_1 + \sum_{i=2}^{2s+1} (s+1-i) e_i + \sum_{j=2s+2}^n (n+1-j) e_j, \\ \lambda_3 &= \sum_{i=1}^{2s} (s+1-i) e_i + (-1-s) e_{2s+1} + \sum_{j=2s+2}^n (n+1-j) e_j. \end{aligned}$$

If $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and if $k = 2s+1$ is an odd positive integer such that $3k \geq 2n+1$ and $k < n$, then we may write:

$$\begin{aligned} \lambda_2 &= (s+1) e_1 + \sum_{i=2}^{2s+1} (s+1-i) e_i + \sum_{j=2s+2}^n (n-j) e_j, \\ \lambda_3 &= \sum_{i=1}^{2s} (s+1-i) e_i + (-1-s) e_{2s+1} + \sum_{j=2s+2}^n (n-j) e_j. \end{aligned}$$

Claim 1 $M_{\Theta^k}(\lambda_3)$ is reducible.

Proof of Claim 1 We apply Jantzen's Criterion of irreducibility (Theorem 2.2.11).

If $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ and if $k = 2s$ is an even positive integer such that $3k > 2n+1$ and $k < n$, then we have

$$\sum_{\substack{\beta \in (\Delta^{\Theta^k})^+ \\ \langle \lambda_3, \beta^\vee \rangle \in \mathbb{N} - \{0\}}} \Upsilon_{\Theta^k}(s_\beta \lambda_3) = \Upsilon_{\Theta^k}(s_{e_1} \lambda_3) = -[M_{\Theta^k}(\lambda_4)] \neq 0.$$

If $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ and if $k = 2s+1$ is an odd positive integer such that $3k \leq 2n$ and $1 < k$, then we may have

$$\begin{aligned} \sum_{\substack{\beta \in (\Delta^{\Theta^k})^+ \\ \langle \lambda_3, \beta^\vee \rangle \in \mathbb{N} - \{0\}}} \Upsilon_{\Theta^k}(s_\beta \lambda_3) &= \Upsilon_{\Theta^k}(s_{e_1} \lambda_3) + \Upsilon_{\Theta^k}(s_{e_1+e_{s+1}} \lambda_3) + \Upsilon_{\Theta^k}(s_{e_1+e_{n-s+1}} \lambda_3) \\ &= [M_{\Theta^k}(\lambda_4)] - [M_{\Theta^k}(\lambda_4)] - [M_{\Theta^k}(\lambda_4)] = -[M_{\Theta^k}(\lambda_4)] \neq 0. \end{aligned}$$

If $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and if $k = 2s+1$ is an odd positive integer such that $3k \geq 2n+1$ and $k < n$, then we have

$$\sum_{\substack{\beta \in (\Delta^{\Theta^k})^+ \\ \langle \lambda_3, \beta^\vee \rangle \in \mathbb{N} - \{0\}}} \Upsilon_{\Theta^k}(s_\beta \lambda_3) = \Upsilon_{\Theta^k}(s_{e_1+e_{n-s+1}} \lambda_3) = -[M_{\Theta^k}(\lambda_4)] \neq 0.$$

Therefore we have Claim 1. \square

Claim 2 There is a non-trivial $U(\mathfrak{g})$ -homomorphism $\varphi : M_{\Theta^k}(\lambda_2) \rightarrow M_{\Theta^k}(\lambda_1)$.

Proof of Claim 2 We remark that neither $\lambda_3 - \lambda_1$ nor $\lambda_3 - \lambda_2$ are contained in \mathbb{Q}^+ . So, Claim 1 implies that $M_{\Theta^k}(\lambda_4) \subseteq M_{\Theta^k}(\lambda_3)$. On the other hand, Zuckerman's duality theorem ([Boe-Collingwood 1985] 4.9, [Collingwood-Shelton 1990] Theorem 1.1, [Gyoja 2000]) implies

$$\dim \operatorname{Hom}_{U(\mathfrak{g})}(M_{\Theta^k}(\lambda_4), M_{\Theta^k}(\lambda_3)) = \dim \operatorname{Hom}_{U(\mathfrak{g})}(M_{\Theta^k}(\lambda_2), M_{\Theta^k}(\lambda_1)).$$

Hence there is a non-trivial $\varphi \in \operatorname{Hom}_{U(\mathfrak{g})}(M_{\Theta^k}(\lambda_2), M_{\Theta^k}(\lambda_1))$. \square

We need:

Claim 3 $\dim(L(\lambda_2)) = \dim \mathfrak{n}_{\Theta^k}$.

Since the multiplicity (the Bernstein degree) of any scalar generalized Verma module is one, Claim 2 and Claim 3 imply $L(\lambda_2)$ is a unique irreducible constituent of $M_{\Theta^k}(\lambda_1) = M_{\Theta^k}[1]$ whose Gelfand-Kirillov dimension is $\dim \mathfrak{n}_{\Theta^k}$. On the other hand, $\dim(M_{\Theta^k}[-1]) = \dim \mathfrak{n}_{\Theta^k}$ and $M_{\Theta^k}[-1]$ is irreducible. Hence, $M_{\Theta^k}[-1] \not\subseteq M_{\Theta^k}[1]$ and we have Lemma 3.7.2. So, we have only to show Claim 3.

Proof of Claim 3

Let $\sigma \in W$ be the longest element (with respect to the length $\ell(\cdot)$) in

$$\{w \in W \mid \lambda_2 \text{ is dominant with respect to } w\Delta^+\}.$$

Then, $\sigma\rho$ and λ_2 are contained in the same closed Weyl chamber and we can regard $L(\lambda_2)$ as a limit of $L(\sigma\rho)$. Namely, $T_{\sigma\rho}^{\lambda_2}(L(\sigma\rho)) \cong L(\lambda_2)$. Since $\dim(L(\sigma\rho)) = \dim(\lambda_2)$, Claim 3 is reduced to the following Claim 4. (Cf. Lemma 2.1.6.)

Claim 4 $\sigma \stackrel{LR}{\sim} w_{\Theta^k} w_0$.

Proof of Claim 4 Claim 4 is obtained by the algorithm described in [Barbasch-Vogan 1982] p171-175.

First, we assume $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ and that k is an odd positive integer such that $3k \leq 2n$ and $1 < k$.

We put $\delta = 10^{-23}$. (In fact, δ can be any real number such that $0 < \delta < \frac{1}{2}$.) Put $\lambda'_2 = \lambda_2 - \delta \omega_k$. We also put $a_i = \langle e_i, \lambda'_2 \rangle$ for $1 \leq i \leq n$. Namely, $\lambda'_2 = \sum_{i=1}^n a_i e_i$. Then, we easily see $|a_1|, \dots, |a_n|$ are distinct. Let b_1, \dots, b_n be positive real numbers such that $b_1 > b_2 > \dots > b_n$ and $\{b_1, \dots, b_n\} = \{|a_1|, \dots, |a_n|\}$. So, there is a permutation $\tau \in \mathfrak{S}_n$ such that $|a_i| = b_{\tau(i)}$ for $1 \leq i \leq n$. For $1 \leq i \leq n$, we put $c_i = \frac{a_i}{|a_i|}(n - \tau(i) + 1)$. Then, we have $\sigma\rho = \sum_{i=1}^n c_i e_n$. Barbasch and Vogan attach σ to the sequence $(c_1, \dots, c_n, -c_n, \dots, -c_1)$ ([Barbasch-Vogan 1982] p173). Applying the Robinson-Schensted algorithm to this sequence, we get a pair of Young tableaux. (These Young tableaux have the same shape.) Remark that the Robinson-Schensted algorithm in [Barbasch-Vogan 1982] is a little bit different from the usual one (see [Barbasch-Vogan 1982] p171). For our purpose, the important information is the shape of these Young tableaux. In order to obtain such a Young diagram, we need not compute c_1, \dots, c_n . In fact, applying the Robinson-Schensted algorithm to the sequence $(a_1, \dots, a_n, -a_n, \dots, -a_1)$ directly, we have the same Young diagram. In this case, the Young diagram corresponds to the partition of $2n$, $(2n - 2k, k + 1, k - 1)$. It corresponds to the symbol $\left(\begin{array}{c} \frac{k-1}{2} \\ n - k + 1 \end{array} \right)$.

On the other hand, $w_{\Theta^k} w_0$ corresponds to the partition of $2n$, $(2n - 2k, k, k)$. It corresponds to the symbol $\left(\begin{array}{c} \frac{k+1}{2} \\ n - k + 1 \end{array} \right)$. Hence, from [Barbasch-Vogan 1982] Theorem 18, we have $\sigma \stackrel{LR}{\sim} w_{\Theta^k} w_0$.

Next, we consider the case of $\mathfrak{g} = \mathfrak{so}(2n + 1, \mathbb{C})$. We define λ'_2 , a_i ($1 \leq i \leq n$), b_i ($1 \leq i \leq n$), $\tau \in \mathfrak{S}_n$, and c_i ($1 \leq i \leq n$) in the same way as the case of $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. In the Barbasch-Vogan algorithm, W is regarded as the Weyl group of the type C_n rather than B_n . So, σ is attached to $(c_1, \dots, c_n, -c_n, \dots, -c_1)$. Again, we may apply the Robinson-Schensted algorithm to the sequence $(a_1, \dots, a_n, -a_n, \dots, -a_1)$ directly and obtain a partition $(k + 1, k - 1, 2n - 2k)$ of $2n$. It corresponds to a symbol $\left(\begin{array}{c} n - k \\ \frac{k}{2} + 1 \end{array} \right)$. On the other hand, $w_{\Theta^k} w_0$ corresponds to the partition of $2n$, $(k, k, 2n - 2k)$. It corresponds to a symbol $\left(\begin{array}{c} n - k \\ \frac{k}{2} \end{array} \right)$. Hence, from [Barbasch-Vogan 1982] Theorem 18, we have $\sigma \stackrel{LR}{\sim} w_{\Theta^k} w_0$.

Finally, we consider the case of $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. In this case, we put $\lambda'_2 = \lambda_2 - \delta \omega_k + \frac{\delta}{2} e_n$. Here, δ is a fixed real number such that $0 < \delta < \frac{1}{2}$. We define a_i ($1 \leq i \leq n$), b_i ($1 \leq i \leq n$), $\tau \in \mathfrak{S}_n$, and c_i ($1 \leq i \leq n$) in the same way as the case of $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. σ is attached to $(c_1, \dots, c_n, -c_n, \dots, -c_1)$. Again, we may apply the Robinson-Schensted algorithm to the sequence $(a_1, \dots, a_n, -a_n, \dots, -a_1)$ directly and obtain a partition $(k + 1, k - 1, 2n - 2k)$ of $2n$. It corresponds to a symbol $\left(\begin{array}{c} n - k \\ 0 \end{array} \right)$. On the other hand, $w_{\Theta^k} w_0$ corresponds to the partition of $2n$, $(k, k, 2n - 2k)$. It corresponds to a symbol $\left(\begin{array}{c} n - k \\ 0 \end{array} \right)$. Hence, from [Barbasch-Vogan 1982] Theorem 18, we have $\sigma \stackrel{LR}{\sim} w_{\Theta^k} w_0$. Q.E.D.

§ 4. Exceptional algebras

As in §3, we write $\Theta^k = \Pi - \{\alpha_k\}$, $\omega_k = \omega_{\alpha_k}$, $c_k = c_{\alpha_k}$, $d_k = d_{\alpha_k}$, etc.

4.1 Even parabolic subalgebras

Let $u \in \mathfrak{g}$ be a nilpotent element. From the Jacobson-Morozov theorem, there is an \mathfrak{sl}_2 -triple (v, h, u) in \mathfrak{g} . Namely, we have $[u, v] = h$, $[h, u] = 2u$, and $[h, v] = -2v$. It is well known that any eigenvalue of the operator $\text{ad}(h) \in \text{End}(\mathfrak{g})$ is an integer. u is called an even nilpotent element if each eigenvalue of $\text{ad}(h)$ is an even number. Put $\mathfrak{p}_u = \sum_{i \geq 0} \{X \in \mathfrak{g} \mid [h, X] = iX\}$.

Definition 4.1.1. *A parabolic subalgebra \mathfrak{p} is called even, if there exists an even nilpotent element u such that $\mathfrak{p} = \mathfrak{p}_u$.*

For an even parabolic subalgebra \mathfrak{p}_u , the Richardson orbit $\mathcal{O}_{\mathfrak{p}_u}$ contains u .

The following result is well-known. For example, it is an easy consequence of [Hesslink 1978] p218 and [Yamashita 1986] Lemma 3.5.

Lemma 4.1.2. *Let \mathfrak{p}_Θ be an even parabolic subalgebra of \mathfrak{g} . Then, the moment map $m_\Theta : T^*X_\Theta \rightarrow \overline{\mathcal{O}_{\mathfrak{p}_\Theta}}$ is birational.*

4.2 G_2

In the regular integral case, homomorphisms between (not necessarily scalar) generalized Verma modules are classified for G_2 in [Boe-Collingwood 1990].

I imagine in the case of G_2 the classification of the homomorphisms between scalar generalized Verma modules is well-known, but I would like to state the result for the completeness.

Let \mathfrak{g} be a simple Lie algebra of the type G_2 . Then, we may write $\Pi = \{\alpha_1, \alpha_2\}$ such that $\langle \alpha_1, \alpha_1 \rangle > \langle \alpha_2, \alpha_2 \rangle$.

Theorem 4.2.1. *Let \mathfrak{g} be a simple Lie algebra of the type G_2 and let k be 1 or 2. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \mathbb{N}$.*

Proof. We see $c_1 = c_2 = \frac{1}{2}$, $d_1 = \frac{3}{2}$, and $d_2 = \frac{5}{2}$. Thus, for $k = 1, 2$, $\rho_{\Theta^k} + t\omega_k$ is integral if and only if $t - \frac{1}{2} \in \mathbb{Z}$. For $k = 1, 2$, it is easy to see that $w_{\Theta^k}w_0 = w_0w_{\Theta^k}$ but they are not a Duflo involution of W . Moreover, we have:

Lemma 4.2.2. *Let k be 1 or 2. If $\rho_{\Theta^k} + t\omega_k$ is singular and integral, then $M_{\Theta^k}[t]$ is irreducible. In particular, $M_{\Theta^k}[\frac{1}{2}]$ is irreducible.*

Proof. \mathfrak{g} has only three special nilpotent orbits, namely the regular nilpotent orbit, the subregular nilpotent orbit, and $\{0\}$. (Cf. [Carter 1985]) Hence, the Gelfand-Kirillov dimension of any infinite-dimensional irreducible constituent of $M_{\Theta^k}[t]$ is $\dim \mathfrak{n}_{\Theta^k} = \text{Dim}(M_{\Theta^k}[t])$, if $\rho_{\Theta^k} + t\omega_k$ is integral. On the other hand, the multiplicity (the Bernstein degree) of $M_{\Theta^k}[t]$ is one. So, if $\rho_{\Theta^k} + t\omega_k$ is integral and singular, $M_{\Theta^k}[t]$ is irreducible. \square

We continue the proof of Theorem 4.2.1. From Lemma 4.2.2 and Lemma 2.2.6, we have $M_{\Theta^k}[-\frac{1}{2} - t] \not\subseteq M_{\Theta^k}[\frac{1}{2} + t]$ for all $t \in \mathbb{N}$.

For $t \in \mathbb{Z}$, $\rho_{\Theta^k} + t\omega_k$ is half-integral and its integral Weyl group is of type $A_2 \times A_2$. This case, we easily see $w_{\Theta^k}w_0$ is a Duflo involution of the integral Weyl group. Since $\rho_{\Theta^k} + t\omega_k$ is

dominant and regular if $t \in \mathbb{N} - \{0\}$, Theorem 1.4.2 implies $M_{\Theta^k}[-1-t] \subseteq M_{\Theta^k}[1+t]$ for all $t \in \mathbb{N}$. Q.E.D.

4.3 F_4

For the simple algebra of the type F_4 , [Boe-Collingwood] treated the regular integral case. The half-integral case is somewhat easier, but I would like to mention the results for the completeness.

We consider the root system Δ for a simple Lie algebra \mathfrak{g} of the type F_4 . (For example, see [Knapp 2002] p691.) We can choose an orthonormal basis e_1, \dots, e_4 of \mathfrak{h}^* such that

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\} \cup \{\pm e_i \mid 1 \leq i \leq 4\} \cup \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\}.$$

We choose a positive system as follows.

$$\Delta^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq 4\} \cup \{e_i \mid 1 \leq i \leq 4\} \cup \left\{ \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \right\}.$$

Put $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$, $\alpha_2 = e_4$, $\alpha_3 = e_3 - e_4$, and $\alpha_4 = e_2 - e_3$. Then, $\Pi = \{\alpha_1, \dots, \alpha_4\}$.

$$1 \quad - \quad 2 \quad \Leftarrow \quad 3 \quad - \quad 4$$

Since $P = Q$ in this case, we have $c_k = \frac{1}{2}$ for $1 \leq k \leq 4$.

The result is:

Theorem 4.3.1. *((1) is due to [Lepowsky 1975a].)*

- (1) $M_{\Theta^1}[-t] \subseteq M_{\Theta^1}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.
- (2) $M_{\Theta^2}[-t] \subseteq M_{\Theta^2}[t]$ if and only if $t \in \mathbb{N}$.
- (3) $M_{\Theta^3}[-t] \subseteq M_{\Theta^3}[t]$ if and only if $t \in \mathbb{N}$.
- (4) $M_{\Theta^4}[-t] \subseteq M_{\Theta^4}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.

Proof. (1) is proved in [Lepowsky 1975a] (Theorem 1.2). So, we consider the other cases.

Since $c_k = \frac{1}{2}$ for $k = 1, 2, 3, 4$, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ implies $t \in \frac{1}{2}\mathbb{N}$.

For each $k = 1, 3, 4$, \mathfrak{p}_{Θ^k} is an even parabolic subalgebra. (Cf. [Carter 1985] p401.) Hence, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ for $t \in \mathbb{N}$ for $k = 1, 3, 4$.

We consider the case of $k = 2$. $\rho_{\Theta^2} + t\omega_2$ is half-integral for $t \in \mathbb{Z}$. The integral root system $\Delta_{\rho_{\Theta^2}}$ for $\rho_{\Theta^2} + t\omega_2$ ($t \in \mathbb{N}$) is of the type $A_1 \times B_3$. Θ^2 corresponds to the set of the long simple roots in the B_2 factor of $\Delta_{\rho_{\Theta^2}}$ and the simple root in the A_1 -factor. Hence, Theorem 3.2.1 (2a), Lemma 1.4.4, and Theorem 1.4.2 imply that $w_{\Theta^2}w_0$ is a Duflo involution in the integral Weyl group. So, Theorem 1.4.2 implies $M_{\Theta^2}[-t] \subseteq M_{\Theta^2}[t]$ for $t \in \mathbb{N}$.

For $k = 2, 3$, we can prove that $M_{\Theta^k}[\frac{1}{2}]$ is irreducible using Jantzen's criterion (Theorem 2.2.11). Hence, Lemma 2.2.6 implies that $M_{\Theta^k}[-\frac{1}{2} - t] \not\subseteq M_{\Theta^k}[\frac{1}{2} + t]$ for $t \in \mathbb{N}$ and $k = 2, 3$.

For $k = 4$, $\rho_{\Theta^4} + (\frac{1}{2} + t)\omega_4$ is half-integral for $t \in \mathbb{N}$. The integral root system $\Delta_{\rho_{\Theta^4}}$ for $\rho_{\Theta^4} + t\omega_4$ ($t \in \mathbb{N}$) is of the type $A_1 \times B_3$. In this case w_{Θ^4} is the non-trivial element of the A_1 -factor of the integral Weyl group. So, it is a Duflo involution in the integral Weyl group and we have $M_{\Theta^4}[-\frac{1}{2} - t] \subseteq M_{\Theta^4}[\frac{1}{2} + t]$ for $t \in \mathbb{N}$. Q.E.D.

4.4 E₆

We consider the root system Δ for a simple Lie algebra \mathfrak{g} of the type E_6 . Put $\kappa = \frac{1}{2\sqrt{3}}$. We can choose an orthonormal basis e_1, \dots, e_6 of \mathfrak{h}^* such that

$$\begin{aligned} \Delta = & \{e_i - e_j \mid 1 \leq i, j \leq 6, i \neq j\} \\ & \cup \left\{ \pm \sum_{i=1}^6 \left(\kappa + (-1)^{n(i)} \frac{1}{2} \right) e_i \mid n(i) = 1 \text{ or } n(i) = 0 \text{ for } 1 \leq i \leq 6, \sum_{i=1}^6 n(i) = 3 \right\} \\ & \cup \left\{ \pm 2\kappa \sum_{i=1}^6 e_i \right\}. \end{aligned}$$

We choose a positive system as follows.

$$\begin{aligned} \Delta^+ = & \{e_i - e_j \mid 1 \leq i < j \leq 6\} \\ & \cup \left\{ \sum_{i=1}^6 \left(\kappa + (-1)^{n(i)} \frac{1}{2} \right) e_i \mid n(i) = 1 \text{ or } n(i) = 0 \text{ for } 1 \leq i \leq 6, \sum_{i=1}^6 n(i) = 3 \right\} \\ & \cup \left\{ 2\kappa \sum_{i=1}^6 e_i \right\}. \end{aligned}$$

Put $\alpha_i = e_i - e_{i+1}$ ($1 \leq i \leq 5$) and $\alpha_6 = \sum_{i=1}^3 (\kappa - \frac{1}{2}) e_i + \sum_{i=4}^6 (\kappa + \frac{1}{2}) e_i$. Then, $\Pi = \{\alpha_1, \dots, \alpha_6\}$.

$$\begin{array}{ccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 5 \\ & & & & | & & & & \\ & & & & 6 & & & & \end{array}$$

In this case, w_0 is not contained in the center of the Weyl group. So, $w_\Theta w_0 = w_0 w_\Theta$ may fail for some $\Theta \subseteq \Pi$. In fact, $w_{\Theta^k} w_0 = w_0 w_{\Theta^k}$ holds for $k = 3, 6$, but it fails for $k = 1, 2, 4, 5$.

Since the Dynkin diagram of E_6 has a symmetry, the cases of Θ^4 and Θ^5 are similar to Θ^2 and Θ^1 , respectively. So, we only consider $\Theta^1, \Theta^2, \Theta^3$, and Θ^6 .

We have:

Theorem 4.4.1. ((1) is due to [Boe 1985].)

- (1) $M_{\Theta^1}[-t] \subseteq M_{\Theta^1}[t]$ if and only if $t = 0$.
- (2) $M_{\Theta^2}[-t] \subseteq M_{\Theta^2}[t]$ if and only if $t = 0$.
- (3) $M_{\Theta^3}[-t] \subseteq M_{\Theta^3}[t]$ if and only if $t \in \mathbb{N}$.
- (4) $M_{\Theta^6}[-t] \subseteq M_{\Theta^6}[t]$ if and only if $t \in \mathbb{N}$.

Proof. (1) is proved in [Boe 1985]. So, we consider the other cases.

First, we prove (2). In this case, we have $\rho^{\Theta^2} = 3\omega_2$. We also have $\rho^{\Theta^2} \in \mathbb{Q}$ and $c_2 = \frac{3}{2}$. Hence, $M_{\Theta^2}[-t] \subseteq M_{\Theta^2}[t]$ implies $t \in \frac{3}{2}\mathbb{N}$. If $t \in 3\mathbb{N}$, then $\rho_{\Theta^2} + t\omega_2$ is integral.

Assume that $t - \frac{3}{2} \in 3\mathbb{N}$. Then, the integral root system of $\rho_{\Theta^2} + t\omega_2$ is of the type $A_1 \times A_5$. The set of simple roots is $\{\alpha_1, \alpha_6, \alpha_3, \alpha_4, \alpha_5, \beta\}$. Here, $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6 = \sum_{i=1}^2 (\kappa + \frac{1}{2}) e_i + \sum_{i=3}^5 (\kappa - \frac{1}{2}) e_i + (\kappa + \frac{1}{2}) e_6$. Among them, α_1 is the simple root in the A_1 -factor.

For $t \in \frac{3}{2}\mathbb{N}$, we denote by $W_{(t)}$ the integral Weyl group of $\rho_{\Theta^2} + t\omega_3$. Let w_0^t be the longest element of the integral Weyl group. We easily see that $w_{\Theta^2}w_0^t$ is not an involution. We also see $\rho_{\Theta^2} + t\omega_3$ is dominant regular for $t \in \frac{3}{2}\mathbb{N} - \{0\}$. Since $w_{\Theta^2}(\rho_{\Theta^2} - t\omega_3) = -(\rho_{\Theta^2} + t\omega_3)$, one of the following two conditions must hold.

- (a) There is no $w \in W_{(t)}$ such that $w(\rho_{\Theta^2} + t\omega_3) = \rho_{\Theta^2} - t\omega_3$.
- (b) $w_{\Theta^2}w_0^t(\rho_{\Theta^2} + t\omega_3) = \rho_{\Theta^2} - t\omega_3$.

If (a) holds, then $M_{\Theta^2}[t]$ has an infinitesimal character different from that of $M_{\Theta^2}[-t]$. Hence, we have $M_{\Theta^2}[-t] \not\subseteq M_{\Theta^2}[t]$.

Assume (b) holds. Then, Theorem 1.4.2 implies that $M_{\Theta^2}[-t] \not\subseteq M_{\Theta^2}[t]$, since $w_{\Theta^2}w_0^t$ is not an involution.

Next, we prove (3) and (4). Let k be either 3 or 6. Then we have $w_{\Theta^k}w_0 = w_0w_{\Theta^k}$ and \mathfrak{p}_k is an even parabolic subalgebra (see [Carter 1985] p402). Hence, Lemma 4.1.2 and Corollary 2.2.9 imply $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ for all $t \in \mathbb{N}$. Since $c_k = \frac{1}{2}$ in these case, we have only to show $M_{\Theta^2}[-\frac{1}{2} - t] \not\subseteq M_{\Theta^2}[\frac{1}{2} + t]$ for all $t \in \mathbb{N}$. In this case, we can check the irreducibility of $M_{\Theta^k}[\frac{1}{2}]$ via Jantzen's criterion (Theorem 2.2.11). So, from Lemma 2.2.6, we have the desired result. We describe the computation briefly.

First, we consider the case of $k = 3$. In this case $d_3 = \frac{7}{2}$ and $\rho_{\Theta^3} + \frac{1}{2}\omega_3$ is integral. We put $\gamma_1 = \sum_{i=1}^6 2\kappa e_i$ and $\gamma_2 = \sum_{i=1}^3 (\kappa + \frac{1}{2}) e_i + \sum_{i=4}^6 (\kappa - \frac{1}{2}) e_i$.

Put $\Xi = \left\{ \beta \in \left(\Delta^{\Theta^3} \right)^+ \mid \langle \rho_{\Theta^3} + \frac{1}{2}\omega_3, \beta^\vee \rangle \in \mathbb{N} - \{0\} \right\}$. Then, $\gamma_1, \gamma_2 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \gamma_2\}$, we can find $\eta \in \Delta_{\Theta^3}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $\Upsilon_{\Theta^3}(s_\beta(\rho_{\Theta^3} + \frac{1}{2}\omega_3)) = 0$. Moreover, we have $\Upsilon_{\Theta^3}(s_{\gamma_1}(\rho_{\Theta^3} + \frac{1}{2}\omega_3)) = -\Upsilon_{\Theta^3}(\rho_{\Theta^3} - \frac{1}{2}\omega_3)$ and $\Upsilon_{\Theta^3}(s_{\gamma_2}(\rho_{\Theta^3} + \frac{1}{2}\omega_3)) = \Upsilon_{\Theta^3}(\rho_{\Theta^3} - \frac{1}{2}\omega_3)$. Hence, we have

$$\sum_{\beta \in \Xi} \Upsilon_{\Theta^3}(s_\beta(\rho_{\Theta^3} + \frac{1}{2}\omega_3)) = 0.$$

This means that Jantzen's criterion is satisfied.

Finally, we consider the case of $k = 6$. In this case $d_6 = \frac{11}{2}$ and $\rho_{\Theta^6} + \frac{1}{2}\omega_6$ is integral. We put $\gamma_1 = \sum_{i=1}^6 2\kappa e_i$ and $\gamma_2 = \sum_{i=1}^6 (\kappa + (-1)^{i-1}\frac{1}{2}) e_i$.

Put $\Xi = \left\{ \beta \in \left(\Delta^{\Theta^6} \right)^+ \mid \langle \rho_{\Theta^6} + \frac{1}{2}\omega_6, \beta^\vee \rangle \in \mathbb{N} - \{0\} \right\}$. Then, $\gamma_1, \gamma_2 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \gamma_2\}$, we can find $\eta \in \Delta_{\Theta^6}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $\Upsilon_{\Theta^6}(s_\beta(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = 0$. Moreover, we have $\Upsilon_{\Theta^6}(s_{\gamma_1}(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = \Upsilon_{\Theta^6}(\rho_{\Theta^6} - \frac{1}{2}\omega_6)$ and $\Upsilon_{\Theta^6}(s_{\gamma_2}(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = -\Upsilon_{\Theta^6}(\rho_{\Theta^6} - \frac{1}{2}\omega_6)$. Hence, we have

$$\sum_{\beta \in \Xi} \Upsilon_{\Theta^6}(s_\beta(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = 0.$$

This means that Jantzen's criterion is satisfied. Q.E.D.

Remark In the case of $k = 6$, the non-existence of the homomorphism is proved in [Boe-Collingwood 1990] for the regular integral case.

4.5 E_7

Let \mathfrak{g} be a simple Lie algebra of the type E_7 .

We fix an orthonormal basis e_1, \dots, e_8 in \mathbb{R}^8 . We identify \mathfrak{h}^* with $\{v \in \mathbb{R}^8 \mid \langle v, e_1 - e_2 \rangle = 0\}$ so that

$$\Delta = \{\pm(e_1 + e_2)\} \cup \{\pm e_i \pm e_j \mid 3 \leq i < j \leq 8\} \\ \cup \left\{ \pm \frac{1}{2} \left(e_1 + e_2 + \sum_{i=3}^8 (-1)^{n(i)} e_i \right) \mid n(i) \text{ is either 0 or 1 for } 3 \leq i \leq 8 \text{ and } \sum_{i=3}^8 n(i) \text{ is even.} \right\}$$

We choose a positive system as follows.

$$\Delta^+ = \{(e_1 + e_2)\} \cup \{e_i \pm e_j \mid 3 \leq i < j \leq 8\} \\ \cup \left\{ \frac{1}{2} \left(e_1 + e_2 + \sum_{i=3}^8 (-1)^{n(i)} e_i \right) \mid n(i) \text{ is either 0 or 1 for } 3 \leq i \leq 8 \text{ and } \sum_{i=3}^8 n(i) \text{ is even.} \right\}$$

Put $\alpha_i = e_{i+2} - e_{i+3}$ for $1 \leq i \leq 5$, $\alpha_6 = e_7 + e_8$, and $\alpha_7 = \frac{1}{2}(e_1 + e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8)$. Then, $\Pi = \{\alpha_1, \dots, \alpha_7\}$ is the set of simple roots in Δ^+ .

$$\begin{array}{cccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 6 & - & 7 \\ & & & & & & \downarrow & & & & \\ & & & & & & 5 & & & & \end{array}$$

We have:

Theorem 4.5.1. (*The case of $k = 1$ is due to [Boe 1985].*)

- (1) Assume $k \in \{1, 3, 5, 6, 7\}$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \mathbb{N}$.
- (2) Assume $k \in \{2, 4\}$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.

Proof. For the simple Lie algebra of the type E_7 , w_0 is contained in the center of the Weyl group. Moreover, all the maximal parabolic subalgebras are even. (See [Carter 1985] p403-404.) So, Lemma 4.1.2 and Corollary 2.2.9 imply that, for all $1 \leq k \leq 8$, $t \in \mathbb{N}$ implies $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$. Hence, we have only to take care of the case of $t - \frac{1}{2} \in \mathbb{N}$.

The case of $k = 1$ is due to [Boe 1985]. In fact $M_{\Theta^1}[-t] \not\subseteq M_{\Theta^1}[t]$ for $t - \frac{1}{2} \in \mathbb{N}$.

For the case $k = 2$, we have $d_2 = 7$ and $\omega_2 \in \mathbb{Q}$. The integral Weyl group for $\rho_{\Theta^2} + t\omega_2$ ($t \in \frac{1}{2} + \mathbb{N}$) is of the type $A_1 \times D_6$. In fact, we have $\Pi_{\rho_{\Theta^2} + t\omega_2} = \{e_3 + e_4\} \cup \Theta^2$. From Theorem 1.4.2 and Theorem 3.2.3, we have $w_{\Theta^2} w_0$ is a Duflo involution of the integral Weyl group for $\rho_{\Theta^2} + t\omega_2$. So, Theorem 1.4.2, Lemma 2.2.6 imply that $M_{\Theta^2}[-t] \subseteq M_{\Theta^2}[t]$ for $t - \frac{1}{2} \in \mathbb{N}$.

For the case $k = 3$, we have $d_3 = 5$ and $\omega_3 = \frac{3}{2}(e_1 + e_2) + e_3 + e_4 + e_5 \notin \mathbb{Q}$. So, we have $c_3 = 1$. Hence, $M_{\Theta^3}[-t] \not\subseteq M_{\Theta^3}[t]$ for $t - \frac{1}{2} \in \mathbb{N}$.

For the case $k = 4$, we have $d_4 = 4$ and $\omega_4 \in \mathbb{Q}$. The integral Weyl group for $\rho_{\Theta^4} + t\omega_4$ ($t \in \frac{1}{2} + \mathbb{N}$) is of the type $A_1 \times D_6$. In fact, we have $\Pi_{\rho_{\Theta^4} + t\omega_4} = \{e_5 + e_6\} \cup \Theta^4$. From Theorem 1.4.2 and Theorem 3.2.3, we have $w_{\Theta^4} w_0$ is a Duflo involution of the integral Weyl group for $\rho_{\Theta^4} + t\omega_4$. So, Theorem 1.4.2, Lemma 2.2.6 imply that $M_{\Theta^4}[-t] \subseteq M_{\Theta^4}[t]$ for $t - \frac{1}{2} \in \mathbb{N}$.

For the case $k = 5$, we have $d_5 = 7$ and $\omega_7 = e_1 + e_2 + \frac{1}{2}(e_3 + e_4 + e_5 + e_6 + e_7 - e_8) \notin \mathbb{Q}$. So, we have $c_5 = 1$. Hence, $M_{\Theta^5}[-t] \not\subseteq M_{\Theta^5}[t]$ for $t - \frac{1}{2} \in \mathbb{N}$.

For the cases $k = 6$ and $k = 7$, we have $d_6 = \frac{11}{2}$ and $d_7 = \frac{17}{2}$. In this case, we can show the irreducibility of $M_{\Theta^k}[\frac{1}{2}]$ ($k = 6, 7$) via Jantzen's criterion (Theorem 2.2.11). So, from Lemma 2.2.6, for $k = 6, 7$, we have $M_{\Theta^k}[-t] \not\subseteq M_{\Theta^k}[t]$ for $t - \frac{1}{2} \in \mathbb{N}$. We describe the computation briefly.

First, we consider the case of $k = 6$. We remark that $P_{\Theta^6}^{++} \cap W(\rho_{\Theta^6} + \frac{1}{2}\omega_6)$ consists of the 3 elements (say $\lambda_1, \lambda_2, \lambda_3$). We put $\lambda_1 = \rho_{\Theta^6} + \frac{1}{2}\omega_6$ and $\lambda_3 = \rho_{\Theta^6} - \frac{1}{2}\omega_6$. The remaining element is $\lambda_2 = \frac{1}{2}e_1 + \frac{1}{2}e_2 + 2e_3 - e_4 - e_6 - 2e_7 - 3e_8$. We put $\gamma_1 = e_1 + e_2$, $\gamma_2 = -\frac{1}{2}e_1 - \frac{1}{2}e_2 + e_3 + 2e_5 - 2e_6 - 3e_7 - e_8$, $\gamma_3 = -\frac{1}{2}e_1 - \frac{1}{2}e_2 + e_3 + 3e_4 + 2e_5 - 2e_6 - e_8$, and $\gamma_4 = \frac{1}{2}e_1 + \frac{1}{2}e_2 + 2e_3 + e_4 - e_6 - 2e_7 - 3e_8$. Put $\Xi = \left\{ \beta \in \left(\Delta^{\Theta^6} \right)^+ \mid \langle \rho_{\Theta^6} + \frac{1}{2}\omega_6, \beta^\vee \rangle \in \mathbb{N} - \{0\} \right\}$. Then, $\gamma_1, \dots, \gamma_4 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \dots, \gamma_4\}$, we can find $\eta \in \Delta_{\Theta^6}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $\Upsilon_{\Theta^6}(s_\beta(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = 0$. Moreover, we have $\Upsilon_{\Theta^6}(s_{\gamma_1}(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = -\Upsilon_{\Theta^6}(\lambda_3)$, $\Upsilon_{\Theta^6}(s_{\gamma_2}(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = \Upsilon_{\Theta^6}(\lambda_3)$, $\Upsilon_{\Theta^6}(s_{\gamma_3}(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = -\Upsilon_{\Theta^6}(\lambda_2)$, and $\Upsilon_{\Theta^6}(s_{\gamma_4}(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = \Upsilon_{\Theta^6}(\lambda_2)$. Hence, we have

$$\sum_{\beta \in \Xi} \Upsilon_{\Theta^6}(s_\beta(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = 0.$$

This means that Jantzen's criterion is satisfied.

First, we consider the case of $k = 7$. We put $\gamma_1 = e_1 + e_2$ and $\gamma_2 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4 + e_5 - e_6 + e_7 + e_8)$.

Put $\Xi = \left\{ \beta \in \left(\Delta^{\Theta^8} \right)^+ \mid \langle \rho_{\Theta^8} + \frac{1}{2}\omega_8, \beta^\vee \rangle \in \mathbb{N} - \{0\} \right\}$. Then, $\gamma_1, \gamma_2 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \gamma_2\}$, we can find $\eta \in \Delta_{\Theta^8}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $\Upsilon_{\Theta^8}(s_\beta(\rho_{\Theta^8} + \frac{1}{2}\omega_8)) = 0$. Moreover, we have $\Upsilon_{\Theta^8}(s_{\gamma_1}(\rho_{\Theta^8} + \frac{1}{2}\omega_8)) = \Upsilon_{\Theta^8}(\rho_{\Theta^8} - \frac{1}{2}\omega_8)$ and $\Upsilon_{\Theta^8}(s_{\gamma_2}(\rho_{\Theta^8} + \frac{1}{2}\omega_8)) = -\Upsilon_{\Theta^8}(\rho_{\Theta^8} - \frac{1}{2}\omega_8)$. Hence, we have

$$\sum_{\beta \in \Xi} \Upsilon_{\Theta^8}(s_\beta(\rho_{\Theta^8} + \frac{1}{2}\omega_8)) = 0.$$

This means that Jantzen's criterion is satisfied.

Q.E.D.

Remark In the case of $k = 7$, the non-existence of the homomorphism is proved in [Boe-Collingwood 1990] for the regular integral case.

4.6 E_8

We fix an orthonormal basis e_1, \dots, e_8 in \mathfrak{h}^* such that

$$\begin{aligned} \Delta = & \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq 8 \} \\ & \cup \left\{ \pm \frac{1}{2} \left(\sum_{i=1}^8 (-1)^{n(i)} e_i \right) \mid n(i) \text{ is either 0 or 1 for } 3 \leq i \leq 8 \text{ and } \sum_{i=3}^8 n(i) \text{ is odd.} \right\} \end{aligned}$$

We choose a positive system as follows.

$$\begin{aligned} \Delta^+ = & \{ e_i \pm e_j \mid 1 \leq i < j \leq 8 \} \\ & \cup \left\{ \frac{1}{2} \left(e_1 + \sum_{i=2}^8 (-1)^{n(i)} e_i \right) \mid n(i) \text{ is either 0 or 1 for } 2 \leq i \leq 8 \text{ and } \sum_{i=2}^8 n(i) \text{ is odd.} \right\} \end{aligned}$$

Put $\alpha_i = e_{i+1} - e_{i+2}$ for $1 \leq i \leq 5$, $\alpha_7 = e_7 + e_8$, and $\alpha_7 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8)$. Then, $\Pi = \{\alpha_1, \dots, \alpha_8\}$ is the set of simple roots in Δ^+ .

$$\begin{array}{cccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 5 & - & 7 & - & 8 \\ & & & & & & & & | & & & & \\ & & & & & & & & 6 & & & & \end{array}$$

We have:

Theorem 4.6.1.

- (1) Assume $k \in \{1, 2, 4, 6, 8\}$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \mathbb{N}$.
- (2) Assume $k \in \{3, 5, 7\}$. Then, $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ if and only if $t \in \frac{1}{2}\mathbb{N}$.

Proof. For any k , we have $w_{\Theta^k}w_0 = w_0w_{\Theta^k}$ since w_0 is contained in the center of W . Hence, Lemma 2.2.3 implies that $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$, only if $t \in \frac{1}{2}\mathbb{N}$.

Next we consider the case of $t \in \mathbb{N}$. For $k \in \{1, 2, 3, 4, 6, 8\}$, \mathfrak{p}_{Θ^k} is even (cf. [Carter 1985] p405-406). In this case, Corollary 2.2.9 and Lemma 4.1.2 imply $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ for $t \in \mathbb{N}$. If $k = 5$, we have $d_5 = \frac{9}{2}$. So, ρ_{Θ^5} is not integral. A basis of integral root system for ρ_{Θ^5} is $\Theta^5 \cup \{e_5 + e_6\}$. We see that the integral root system is of the type $A_1 \times E_7$. So, the problem is reduced to the case of $k = 3$ in the type E_7 . Hence, $M_{\Theta^5}[-t] \subseteq M_{\Theta^5}[t]$ for $t \in \mathbb{N}$ such that $\rho_{\Theta^5} + t\omega_5$ is dominant and regular. From Lemma 2.2.6, we have $M_{\Theta^5}[-t] \subseteq M_{\Theta^5}[t]$ for all $t \in \mathbb{N}$. The case of $k = 7$ is similar to the case of $k = 5$. This time, a basis of the integral Weyl group of ρ_{Θ^7} is $\Theta^7 \cup \{\frac{1}{2}(e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7 + e_8)\}$ and the integral root system is of the type $A_1 \times E_7$. Hence, the problem is reduced to the case of $k = 5$ in the type E_7 .

Next, we consider the case of $t \in \frac{1}{2} + \mathbb{N}$.

First, we consider the case of $k = 3$. In this case, $\rho_{\Theta^3} + t\omega_3$ is not integral for $t \in \frac{1}{2} + \mathbb{N}$. A basis of integral root system for $\rho_{\Theta^3} + \frac{1}{2}\omega_3$ is $\Theta^3 \cup \{e_3 + e_4\}$ and the integral root system is of the type D_8 . This time, the problem is reduced to the case of $k = 3$ in the type D_8 .

Next, we consider the case of $k = 5, 7$. In this case, $\rho_{\Theta^k} + t\omega_k$ is integral for $t \in \frac{1}{2} + \mathbb{N}$. The special representation corresponding to a Richardson orbit can be constructed as a MacDonald representation ([MacDonald 1972]). If $k = 5$ (resp. $k = 7$), then the number of positive roots in the Levi part of \mathfrak{p}_{Θ^k} is 14 (resp. 22). Hence, the special representation occurs as component of $S^{14}(\mathfrak{h}^*)$ (resp. $S^{22}(\mathfrak{h}^*)$) but not of $S^d(\mathfrak{h}^*)$ for $d > 14$ (resp. $d > 22$). Hence, from [Carter 1985] p417-418, we see the family of the special representation consists of a single element. Hence, Corollary 2.1.4, Lemma 2.2.6 and Lemma 2.2.7 imply that $M_{\Theta^k}[-t] \subseteq M_{\Theta^k}[t]$ for $t \in \frac{1}{2} + \mathbb{N}$.

If $k = 1, 2, 4, 6, 8$, we can prove that $M_{\Theta^k}[\frac{1}{2}]$ is irreducible via Jantzen's criterion. So, we have $M_{\Theta^k}[-t] \not\subseteq M_{\Theta^k}[t]$ for $t \in \frac{1}{2} + \mathbb{N}$. We describe the computation briefly.

First, we consider the case of $k = 1$. In this case $d_1 = \frac{29}{2}$ and $\rho_{\Theta^1} + \frac{1}{2}\omega_1$ is integral. We put $\gamma_1 = e_1 + e_2$ and $\gamma_2 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4 + e_5 - e_6 + e_7 - e_8)$.

Put $\Xi = \left\{ \beta \in \left(\Delta^{\Theta^1} \right)^+ \mid \langle \rho_{\Theta^1} + \frac{1}{2}\omega_1, \beta^\vee \rangle \in \mathbb{N} - \{0\} \right\}$. Then, $\gamma_1, \gamma_2 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \gamma_2\}$,

we can find $\eta \in \Delta_{\Theta^1}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $\Upsilon_{\Theta^1}(s_\beta(\rho_{\Theta^1} + \frac{1}{2}\omega_1)) = 0$. Moreover, we have $\Upsilon_{\Theta^1}(s_{\gamma_1}(\rho_{\Theta^1} + \frac{1}{2}\omega_1)) = \Upsilon_{\Theta^1}(\rho_{\Theta^1} - \frac{1}{2}\omega_1)$ and $\Upsilon_{\Theta^1}(s_{\gamma_2}(\rho_{\Theta^1} + \frac{1}{2}\omega_1)) = -\Upsilon_{\Theta^1}(\rho_{\Theta^1} - \frac{1}{2}\omega_1)$. Hence, we have

$$\sum_{\beta \in \Xi} \Upsilon_{\Theta^1}(s_\beta(\rho_{\Theta^1} + \frac{1}{2}\omega_1)) = 0.$$

This means that Jantzen's criterion is satisfied.

Next, we consider the case of $k = 2$. In this case $d_2 = \frac{19}{2}$ and $\rho_{\Theta^2} + \frac{1}{2}\omega_2$ is integral. We remark that $P_{\Theta^k}^{++} \cap W(\rho_{\Theta^2} + \frac{1}{2}\omega_2)$ consists of the 3 elements (say $\lambda_1, \lambda_2, \lambda_3$). We put $\lambda_1 = \rho_{\Theta^2} + \frac{1}{2}\omega_2$ and $\lambda_3 = \rho_{\Theta^2} - \frac{1}{2}\omega_2$. The remaining element is $\lambda_2 = 4e_1 - 3e_2 - 5e_3 + 4e_4 + 3e_5 + 2e_6 + e_7$. We put $\gamma_1 = e_1 + e_2$, $\gamma_2 = e_1 + e_3$, $\gamma_3 = e_1 - e_6$, and $\gamma_4 = \frac{1}{2}(e_1 + e_2 - e_3 + e_4 - e_5 - e_6 + e_7 + e_8)$. Put $\Xi = \left\{ \beta \in \left(\Delta^{\Theta^2} \right)^+ \mid \langle \rho_{\Theta^2} + \frac{1}{2}\omega_2, \beta^\vee \rangle \in \mathbb{N} - \{0\} \right\}$. Then, $\gamma_1, \dots, \gamma_4 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \dots, \gamma_4\}$, we can find $\eta \in \Delta_{\Theta^2}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $\Upsilon_{\Theta^2}(s_\beta(\rho_{\Theta^2} + \frac{1}{2}\omega_2)) = 0$. Moreover, we have $\Upsilon_{\Theta^2}(s_{\gamma_1}(\rho_{\Theta^2} + \frac{1}{2}\omega_2)) = -\Upsilon_{\Theta^2}(\lambda_3)$, $\Upsilon_{\Theta^2}(s_{\gamma_2}(\rho_{\Theta^2} + \frac{1}{2}\omega_2)) = \Upsilon_{\Theta^2}(\lambda_2)$, $\Upsilon_{\Theta^2}(s_{\gamma_3}(\rho_{\Theta^2} + \frac{1}{2}\omega_2)) = \Upsilon_{\Theta^2}(\lambda_3)$, and $\Upsilon_{\Theta^2}(s_{\gamma_4}(\rho_{\Theta^2} + \frac{1}{2}\omega_2)) = -\Upsilon_{\Theta^2}(\lambda_2)$. Hence, we have

$$\sum_{\beta \in \Xi} \Upsilon_{\Theta^2}(s_\beta(\rho_{\Theta^2} + \frac{1}{2}\omega_2)) = 0.$$

This means that Jantzen's criterion is satisfied.

Next, we consider the case of $k = 4$. In this case $d_4 = \frac{11}{2}$ and $\rho_{\Theta^4} + \frac{1}{2}\omega_4$ is integral. We remark that $P_{\Theta^k}^{++} \cap W(\rho_{\Theta^4} + \frac{1}{2}\omega_4)$ consists of the 7 elements $\lambda_1, \dots, \lambda_7$. They are characterized as follows. $\langle \lambda_1, \omega_4 \rangle = 10$, $\langle \lambda_2, \omega_4 \rangle = 5$, $\langle \lambda_3, \omega_4 \rangle = 2$, $\langle \lambda_4, \omega_4 \rangle = 0$, $\langle \lambda_5, \omega_4 \rangle = -2$, $\langle \lambda_6, \omega_4 \rangle = -5$, and $\langle \lambda_7, \omega_4 \rangle = -10$. We have $\lambda_1 = \rho_{\Theta^4} + \frac{1}{2}\omega_4$ and $\lambda_7 = \rho_{\Theta^4} - \frac{1}{2}\omega_4$. We put $\gamma_1 = e_1 + e_2$, $\gamma_2 = e_1 + e_3$, $\gamma_3 = e_1 + e_4$, $\gamma_4 = e_1 + e_5$, $\gamma_5 = e_1 + e_6$, $\gamma_6 = e_1 - e_5$, $\gamma_7 = e_1 - e_7$, $\gamma_8 = e_1 - e_8$, $\gamma_9 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4 - e_5 + e_6 + e_7 - e_8)$, $\gamma_{10} = \frac{1}{2}(e_1 + e_2 - e_3 - e_4 - e_5 + e_6 + e_7 + e_8)$, $\gamma_{11} = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 - e_5 + e_6 + e_7 + e_8)$, and $\gamma_{12} = \frac{1}{2}(e_1 + e_2 + e_3 - e_4 - e_5 + e_6 - e_7 + e_8)$.

Put $\Xi = \left\{ \beta \in \left(\Delta^{\Theta^4} \right)^+ \mid \langle \rho_{\Theta^4} + \frac{1}{2}\omega_4, \beta^\vee \rangle \in \mathbb{N} - \{0\} \right\}$. Then, $\gamma_1, \dots, \gamma_{12} \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \dots, \gamma_{12}\}$, we can find $\eta \in \Delta_{\Theta^4}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $\Upsilon_{\Theta^4}(s_\beta(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = 0$. Moreover, we have $\Upsilon_{\Theta^4}(s_{\gamma_1}(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = -\Upsilon_{\Theta^4}(\lambda_7)$, $\Upsilon_{\Theta^4}(s_{\gamma_2}(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = \Upsilon_{\Theta^4}(\lambda_6)$, $\Upsilon_{\Theta^4}(s_{\gamma_3}(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = -\Upsilon_{\Theta^4}(\lambda_4)$, $\Upsilon_{\Theta^4}(s_{\gamma_4}(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = \Upsilon_{\Theta^4}(\lambda_2)$, $\Upsilon_{\Theta^4}(s_{\gamma_5}(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = \Upsilon_{\Theta^4}(\lambda_7)$, $\Upsilon_{\Theta^4}(s_{\gamma_6}(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = -\Upsilon_{\Theta^4}(\lambda_6)$, $\Upsilon_{\Theta^4}(s_{\gamma_7}(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = \Upsilon_{\Theta^4}(\lambda_3)$, $\Upsilon_{\Theta^4}(s_{\gamma_8}(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = -\Upsilon_{\Theta^4}(\lambda_5)$, $\Upsilon_{\Theta^4}(s_{\gamma_9}(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = \Upsilon_{\Theta^4}(\lambda_4)$, $\Upsilon_{\Theta^4}(s_{\gamma_{10}}(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = -\Upsilon_{\Theta^4}(\lambda_2)$, $\Upsilon_{\Theta^4}(s_{\gamma_{11}}(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = \Upsilon_{\Theta^4}(\lambda_5)$, and $\Upsilon_{\Theta^4}(s_{\gamma_{12}}(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = -\Upsilon_{\Theta^4}(\lambda_3)$. Hence, we have

$$\sum_{\beta \in \Xi} \Upsilon_{\Theta^4}(s_\beta(\rho_{\Theta^4} + \frac{1}{2}\omega_4)) = 0.$$

This means that Jantzen's criterion is satisfied.

Next, we consider the case of $k = 8$. In this case $d_1 = \frac{23}{2}$ and $\rho_{\Theta^1} + \frac{1}{2}\omega_8$ is integral. We put $\gamma_1 = e_1 + e_7$ and $\gamma_2 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4 - e_5 + e_6 + e_7 - e_8)$.

Put $\Xi = \left\{ \beta \in \left(\Delta^{\Theta^8} \right)^+ \mid \langle \rho_{\Theta^8} + \frac{1}{2}\omega_8, \beta^\vee \rangle \in \mathbb{N} - \{0\} \right\}$. Then, $\gamma_1, \gamma_2 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \gamma_2\}$, we can find $\eta \in \Delta_{\Theta^8}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $\Upsilon_{\Theta^8}(s_\beta(\rho_{\Theta^8} + \frac{1}{2}\omega_8)) = 0$. Moreover, we have $\Upsilon_{\Theta^8}(s_{\gamma_1}(\rho_{\Theta^8} + \frac{1}{2}\omega_8)) = \Upsilon_{\Theta^8}(\rho_{\Theta^8} - \frac{1}{2}\omega_8)$ and $\Upsilon_{\Theta^8}(s_{\gamma_2}(\rho_{\Theta^8} + \frac{1}{2}\omega_8)) = -\Upsilon_{\Theta^8}(\rho_{\Theta^8} - \frac{1}{2}\omega_8)$. Hence, we have

$$\sum_{\beta \in \Xi} \Upsilon_{\Theta^8}(s_\beta(\rho_{\Theta^8} + \frac{1}{2}\omega_8)) = 0.$$

This means that Jantzen's criterion is satisfied.

Finally, we consider the case $k = 6$. In this case, we choose the following basis of the root system in order to make computation easier. $\alpha_1 = e_7 - e_8$, $\alpha_2 = e_6 - e_7$, $\alpha_3 = e_5 - e_6$, $\alpha_4 = e_4 - e_5$, $\alpha_5 = e_3 - e_4$, $\alpha_6 = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6 + e_7 + e_8)$, $\alpha_7 = e_2 - e_3$, and $\alpha_8 = e_1 - e_2$. In this case $d_6 = \frac{17}{2}$ and $\rho_{\Theta^6} + \frac{1}{2}\omega_6$ is integral. We put $\gamma_1 = e_1 + e_5$ and $\gamma_2 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 - e_8)$. Put $\Xi = \left\{ \beta \in \left(\Delta^{\Theta^6} \right)^+ \mid \langle \rho_{\Theta^6} + \frac{1}{2}\omega_6, \beta^\vee \rangle \in \mathbb{N} - \{0\} \right\}$. Then, $\gamma_1, \gamma_2 \in \Xi$. For $\beta \in \Xi - \{\gamma_1, \gamma_2\}$, we can find $\eta \in \Delta_{\Theta^6}$ such that $\langle \beta, \eta \rangle = 0$. Hence, we have $\Upsilon_{\Theta^6}(s_\beta(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = 0$. Moreover, we have $\Upsilon_{\Theta^6}(s_{\gamma_1}(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = \Upsilon_{\Theta^6}(\rho_{\Theta^6} - \frac{1}{2}\omega_6)$ and $\Upsilon_{\Theta^6}(s_{\gamma_2}(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = -\Upsilon_{\Theta^6}(\rho_{\Theta^6} - \frac{1}{2}\omega_6)$. Hence, we have

$$\sum_{\beta \in \Xi} \Upsilon_{\Theta^6}(s_\beta(\rho_{\Theta^6} + \frac{1}{2}\omega_6)) = 0.$$

This means that Jantzen's criterion is satisfied. Q.E.D.

Remark In the case of $k = 1$, the non-existence of the homomorphism is proved in [Boe-Collingwood 1990] for the regular integral case.

§ 5. Elementary homomorphisms

Here, we explain that we can construct homomorphisms in the setting of general parabolic subalgebras from the case of maximal parabolic subalgebras.

5.1 A comparison result

Here, we review some notion in [Matumoto 1993] §3. Hereafter, \mathfrak{g} means a reductive Lie algebra over \mathbb{C} and retain the notations in §1. We fix a subset Θ of Π . For $\alpha \in \Delta$, we put

$$\begin{aligned} \Delta(\alpha) &= \{ \beta \in \Delta \mid \exists c \in \mathbb{R} \ \beta|_{\mathfrak{a}_\Theta} = c\alpha|_{\mathfrak{a}_\Theta} \}, \\ \Delta^+(\alpha) &= \Delta(\alpha) \cap \Delta^+, \\ U_\alpha &= \mathbb{C}S + \mathbb{C}\alpha \subseteq \mathfrak{h}^*. \end{aligned}$$

Then $(U_\alpha, \Delta(\alpha), \langle \cdot, \cdot \rangle)$ is a subroot system of $(\mathfrak{h}^*, \Delta, \langle \cdot, \cdot \rangle)$. The set of simple roots for $\Delta^+(\alpha)$ is denoted by $\Pi(\alpha)$. If $\alpha|_{\mathfrak{a}_\Theta} \neq 0$, then there exists a unique $\tilde{\alpha} \in \Delta^+$ such that $\Pi(\alpha) = \Theta \cup \{\tilde{\alpha}\}$. If $\alpha \in \Delta$ satisfies $\alpha|_{\mathfrak{a}_\Theta} \neq 0$ and $\alpha = \tilde{\alpha}$, then we call α Θ -reduced. For $\alpha \in \Delta^+$, we denote by $W_\Theta(\alpha)$ the Weyl group of $(\mathfrak{h}^*, \Delta(\alpha))$. Clearly, $W_\Theta \subseteq W_\Theta(\alpha) \subseteq W$. We denote by w^α the longest element of $W_\Theta(\alpha)$. We call $\alpha \in \Delta$ Θ -acceptable if $w^\alpha w_\Theta = w_\Theta w^\alpha$. We denote by Δ_r^Θ the set of Θ -reduced Θ -acceptable roots. Put $(\Delta_r^\Theta)^+ = \Delta^+ \cap \Delta_r^\Theta$. For $\alpha \in \Delta_r^\Theta$, we define

$$\sigma_\alpha = w^\alpha w_\Theta = w_\Theta w^\alpha.$$

Clearly, $\sigma_\alpha^2 = 1$. For $\alpha \in \Delta$, we put

$$V_\alpha = \{ \lambda \in \mathfrak{a}_\Theta^* \mid \langle \lambda, \alpha \rangle = 0 \}.$$

We denote by $\omega_\alpha \in \mathfrak{a}_\Theta^* \subseteq \mathfrak{h}^*$ the fundamental weight for α with respect to the basis $\Pi(\alpha) = \Theta \cup \{\alpha\}$. Namely ω_α satisfies that $\langle \omega_\alpha, \beta \rangle = 0$ for $\beta \in \Theta$, $\langle \beta, \tilde{\alpha} \rangle = 1$, and $\omega_\alpha|_{\mathfrak{h} \cap \mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Here, $\mathfrak{c}(\mathfrak{g}(\alpha))$ is the center of $\mathfrak{g}(\alpha)$. We see that there is some positive real number a such that $\omega_\alpha = a\alpha|_{\mathfrak{a}_\Theta}$, since $\alpha|_{\mathfrak{h} \cap \mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Hence, we have $V_\alpha = \{ \lambda \in \mathfrak{a}_\Theta^* \mid \langle \lambda, \omega_\alpha \rangle = 0 \}$.

We can easily see:

Lemma 5.1.1. *Let $\alpha \in \Delta_r^\Theta$. Then, we have*

- (1) σ_α preserves \mathfrak{a}_Θ^* .
- (2) $\sigma_\alpha \in W(\Theta)$. In particular, $\sigma_\alpha \rho_\Theta = \rho_\Theta$.
- (3) $\sigma_\alpha \omega_\alpha = -\omega_\alpha$.
- (4) $\sigma_\alpha|_{\mathfrak{a}_\Theta}$ is the reflection with respect to V_α .

For $\alpha \in (\Delta_r^\Theta)^+$, we define

$$\mathfrak{g}(\alpha) = \mathfrak{h} + \sum_{\beta \in \Delta(\alpha)} \mathfrak{g}_\beta, \quad \mathfrak{p}_\Theta(\alpha) = \mathfrak{g}(\alpha) \cap \mathfrak{p}_\Theta.$$

Then, $\mathfrak{g}(\alpha)$ is a reductive Lie subalgebra of \mathfrak{g} whose root system is $\Delta(\alpha)$ and $\mathfrak{p}_\Theta(\alpha)$ is a maximal parabolic subalgebra of $\mathfrak{g}(\alpha)$.

Put $\rho(\alpha) = \frac{1}{2} \sum_{\beta \in \Delta^+(\alpha)} \beta$. For $\nu \in \mathfrak{a}_\Theta^*$, we denote by \mathbb{C}_ν the one-dimensional $U(\mathfrak{p}_\Theta(\alpha))$ -module corresponding to ν . For $\nu \in \mathfrak{a}_\Theta^*$ we define a generalized Verma module for $\mathfrak{g}(\alpha)$ as follows.

$$M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + \nu) = U(\mathfrak{g}(\alpha)) \otimes_{U(\mathfrak{p}_\Theta(\alpha))} \mathbb{C}_{\nu - \rho(\alpha)}.$$

Then, we have:

Theorem 5.1.2. *Let ν be an arbitrary element in V_α and let c be either 1 or $\frac{1}{2}$. Assume that $M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta - nc\omega_\alpha) \subseteq M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + nc\omega_\alpha)$ for all $n \in \mathbb{N}$. Then, we have $M_\Theta(\rho_\Theta + \nu - nc\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + \nu + nc\omega_\alpha)$ for all $n \in \mathbb{N}$. (We call the above homomorphism of $M_\Theta(\rho_\Theta + \nu - nc\omega_\alpha)$ into $M_\Theta(\rho_\Theta + \nu + nc\omega_\alpha)$ an elementary homomorphism.)*

Proof. Assume that $M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta - nc\omega_\alpha) \subseteq M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + nc\omega_\alpha)$ for all $n \in \mathbb{N}$. Remark that $\sigma_\alpha(\rho_\Theta + nc\omega_\alpha) = \rho_\Theta - nc\omega_\alpha$. From Theorem 1.4.2, this implies that σ_α is a Duflo involution of the integral Weyl group $W_{\rho_\Theta + nc\omega_\alpha}$ for a sufficiently large n .

Put $Q_{\alpha,n} = \{\nu \in V_\alpha \mid \Delta(\alpha)_{\rho_\Theta + nc\omega_\alpha} = \Delta_{\rho_\Theta + \nu + nc\omega_\alpha}\}$. From Theorem 1.4.2, for sufficiently large $n \in \mathbb{N}$ and $\nu \in Q_{\alpha,n}$, we have $M_\Theta(\rho_\Theta + \nu - nc\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + \nu + nc\omega_\alpha)$, since we have $\sigma_\alpha(\rho_\Theta + \nu + nc\omega_\alpha) = \rho_\Theta + \nu - nc\omega_\alpha$. We easily see $\nu - nc\omega_\alpha$ is strongly Θ -antidominant for all $n \in \mathbb{N}$. Applying Lemma 1.5.1 and the exactness of the translation functor, we can remove the extra assumption that n is sufficiently large.

On the other hand $Q_{\alpha,n}$ is Zarisky dense in V_α (Cf. [Matumoto 1993] Lemma 3.2.2 (1)). Moreover, we can prove that $\{\nu \in \mathfrak{a}_\Theta^* \mid M_\Theta(\rho_\Theta + \nu - \mu) \subseteq M_\Theta(\rho_\Theta + \nu)\}$ is Zarisky closed in \mathfrak{a}_Θ^* for each $\mu \in \mathfrak{a}_\Theta^*$ in the same way as [Lepowsky 1975b] Lemma 5.4. Hence, for each $\nu \in V_\alpha$ and each $n \in \mathbb{N}$, we have $M_\Theta(\rho_\Theta + \nu - nc\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + \nu + nc\omega_\alpha)$. Q.E.D.

Remark Taking this opportunity, I would like to fix an error in [Matumoto 1993]. In page 269 line 18, the definitions of $\mathfrak{g}(\alpha, c)$ and $\mathfrak{p}_S(\alpha, c)$ are incorrect. $\mathfrak{g}(\alpha, c)$ should be an abstract reductive Lie algebra associated with the pair $(\mathfrak{h}, \Delta_{\alpha,c})$. $\mathfrak{p}_S(\alpha, c)$ should be the standard parabolic subalgebra corresponding to Θ . Since $\Delta_{\alpha,c}$ need not be closed under the addition in Δ , $\mathfrak{g}(\alpha, c)$ need not be a subalgebra of \mathfrak{g} .

5.2 C_n case

As an example, we describe elementary homomorphisms in the C_n case. Let $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. We use the notation in the root system in 3.1.

Let $\kappa = (k_1, \dots, k_s)$ be a finite sequence of positive integers such that $k_1 + \dots + k_s \leq n$. We put $k_i^* = k_1 + \dots + k_i$ for $1 \leq i \leq s$ and $k_0^* = 0$. We define a subset Θ^κ of Π as follows.

$$\Theta^\kappa = \begin{cases} \Pi - \{e_{k_i^*} - e_{k_{i+1}^*} \mid 1 \leq i \leq s\} & \text{if } k_s^* < n, \\ \Pi - (\{e_{k_i^*} - e_{k_{i+1}^*} \mid 1 \leq i \leq s-1\} \cup \{2e_n\}) & \text{if } k_s^* = n \end{cases}.$$

Then the corresponding standard Levi subalgebra $\mathfrak{l}_{\Theta^\kappa}$ is isomorphic to $\mathfrak{gl}(k_1, \mathbb{C}) \oplus \mathfrak{gl}(k_2, \mathbb{C}) \oplus \dots \oplus \mathfrak{gl}(k_s, \mathbb{C}) \oplus \mathfrak{sp}(n - k_s^*, \mathbb{C})$. Here, we regard $\mathfrak{sp}(0, \mathbb{C})$ as a trivial Lie algebra $\{0\}$. Obviously any proper subset of Π is written as the form of Θ^κ .

We put $a_i = \sum_{j=1}^{k_i} e_{k_{i-1}^* + j}$ ($1 \leq i \leq s$). Then, a_1, \dots, a_s form a basis of $\mathfrak{a}_{\Theta^\kappa}^*$. We write $M_{\Theta^\kappa}[t_1, \dots, t_s]$ for $M_{\Theta^\kappa}(\rho_{\Theta^\kappa} + t_1 a_1 + \dots + t_s a_s)$ for $t_1, \dots, t_s \in \mathbb{C}$.

We easily have:

Lemma 5.2.1.

(1) If $k_s^* < n$, then

$$(\Delta_r^{\Theta^\kappa})^+ = \{e_{k_i^*} - e_{k_{j+1}^*} \mid 1 \leq i \leq j < s, k_i = k_{j+1}\} \cup \{e_{k_i^*} - e_{k_s^* + 1} \mid 1 \leq i \leq s\}$$

(2) If $k_s^* = n$, then

$$(\Delta_r^{\Theta^\kappa})^+ = \{e_{k_i^*} - e_{k_{j+1}^*} \mid 1 \leq i \leq j < s, k_i = k_{j+1}\} \cup \{2e_{k_i^*} \mid 1 \leq i \leq s\}$$

Combining [Boe 1985] 4.4 Theorem, Theorem 3.2.2, and Theorem 5.1.2, we have

Proposition 5.2.2. (1) Let $1 \leq p < q \leq s$ be such that $k_i = k_j$. If $t_p - t_q \in \mathbb{N}$, we have

$$M_{\Theta^\kappa} \left(\rho_{\Theta^\kappa} + \sum_{\substack{1 \leq i \leq s \\ i \neq p, q}} t_i a_i + t_q a_p + t_p a_q \right) \subseteq M_{\Theta^\kappa} \left(\rho_{\Theta^\kappa} + \sum_{1 \leq i \leq s} t_i a_i \right).$$

(2) Let $1 \leq p \leq q$ be such that $3k_p > 2(k_p + n - k_s^*)$. If $t_p \in \mathbb{N}$, we have

$$M_{\Theta^\kappa} \left(\rho_{\Theta^\kappa} + \sum_{\substack{1 \leq i \leq s \\ i \neq p}} t_i a_i - t_p a_p \right) \subseteq M_{\Theta^\kappa} \left(\rho_{\Theta^\kappa} + \sum_{1 \leq i \leq s} t_i a_i \right).$$

(3) Let $1 \leq p \leq q$ be such that $3k_p \leq 2(k_p + n - k_s^*)$ and k_p is even. If $t_p \in \frac{1}{2}\mathbb{N}$, we have

$$M_{\Theta^\kappa} \left(\rho_{\Theta^\kappa} + \sum_{\substack{1 \leq i \leq s \\ i \neq p}} t_i a_i - t_p a_p \right) \subseteq M_{\Theta^\kappa} \left(\rho_{\Theta^\kappa} + \sum_{1 \leq i \leq s} t_i a_i \right).$$

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